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AFFECT AND META-AFFECT IN MATHEMATICAL PROBLEM SOLVING: A REPRESENTATIONAL PERSPECTIVE

ABSTRACT. We discuss a research-based theoretical framework based on affect as an internal representational system. Key ideas include the concepts of meta-affect and affective structures, and the constructs of mathematical intimacy and mathematical integrity. We understand these as fundamental to powerful mathematical problem solving, and deserving of closer attention by educators. In a study of elementary school children we characterize some features of emotional states inferred from individual problem solving behavior, including interactions between affect and cognition. We describe our methodology, illustrating theoretical ideas with brief qualitative examples from a longitudinal series of task-based interviews.

KEY WORDS: affect and cognition, affective structures, attitudes, beliefs, emotions, mathematical integrity, mathematical intimacy, mathematical problem solving, meta-affect, representational systems, values

1. INTRODUCTION

Research on mathematical learning and problem solving traditionally highlights cognition, affording less attention to affect or cognitive-affective interactions. One reason may be that mathematics, unlike the humanities, music, or the arts, is commonly understood as “purely rational”, with emotion playing no role. Naturally, teachers try to motivate students by making mathematics enjoyable; but learning mathematics is viewed as essentially cognitive. Another reason may be the methodological difficulty in designing and carrying out reliable empirical studies of affect. Such barriers may stem – at least in part – from the absence of a precise, shared language for describing mathematical affect, within a theoretical framework permitting its systematic study.

But mathematics education researchers are increasingly treating the affective domain seriously, while cognitive scientists skeptical of a more narrow ‘cognitivism’ have stressed the importance of theories integrating cognition, emotion, and motivation (McLeod and Adams, 1994; Dai and Sternberg, 2004). The present article reflects the authors’ 15-year collaboration on a theoretical framework for affect in the context of individual mathematical problem solving (DeBellis, 1996, 1998; DeBellis and Goldin, 1997, 1999; Goldin, 2000, 2002).

We see human affect functioning as an *internal representational system* exchanging information with cognitive systems. *Meta-affect* is a construct central to our theory, standing in relation to affect much as metacognition stands in relation to cognition, and powerfully transforming individuals' emotional feelings. *Mathematical intimacy* refers to deep, vulnerable emotional engagement an individual may have with mathematics. *Mathematical integrity* refers to the individual's fundamental commitment to mathematical truth, search for mathematical understanding, or moral character guiding mathematical study. Taken together, these constructs influence the nature of learning and the depth of knowledge attained.

We seek to infer (internal) affect from (external, observable) behavior in individual children's mathematical problem solving during structured task-based interviews (Goldin et al., 1993; Goldin, 1998b). Here we outline our developing theoretical framework, describe our methodology, illustrate the concepts of meta-affect, mathematical intimacy, and mathematical integrity with interview excerpts, and discuss why the more limited information available for 'Frank' does not accommodate this level of empirical study.

In calling our framework "research based", we do not mean to imply that our constructs have been validated or our conjectures demonstrated in our empirical study. Rather, our research provides illustrative examples for each of our major constructs, and our analysis of affect in specific episodes during task-based problem solving lends plausibility, in a preliminary way, to our main conjectures.

2. ASPECTS OF AFFECT AS A SYSTEM OF REPRESENTATION

2.1. *Affect as representation*

An earlier model for mathematical problem-solving competency is based on five kinds of internal, mutually interacting systems of representation (Goldin, 1998a; Goldin and Kaput, 1996): (a) verbal/syntactic systems, comprising natural language, grammar, and syntax; (b) imagistic systems, including internal visual/spatial, auditory/rhythmic, and tactile/kinesthetic encoding, (c) formal notational systems, e.g. numeration systems, notations for arithmetic, algebra, and calculus, Cartesian graphs, etc.; (d) a system of planning and executive control, governing heuristic and strategic decision-making during problem solving; and (e) an affective system, involving emotions, attitudes, beliefs, morals, values, and ethics. Such representational systems are very complex, developing in individuals through stages over time. Each involves configurations that can stand in symbolic relationship to other configurations – i.e., the configurations *encode meaning*

for the person, and they can be externalized to *communicate meaning* to others.

Representational configurations in each system occur within higher structures – e.g., words and phrases within the syntactic structure of natural language. Moreover, competency structures involve all the different kinds of representational systems, which thus interact continually with each other during problem solving.

Affect includes changing states of emotional feeling during mathematical problem solving (local affect) – feelings of which the individual may be consciously aware, as well as unconscious or preconscious emotional states (Damasio, 1994). It also includes more stable, longer-term constructs (global affect), which establish contexts for local affect and which local affect can influence. Our hypothesis is that affect is fundamentally *representational*, rather than a system of mostly involuntary, physiological side-effects of cognition.

This view – basically a conjecture, but with empirical evidence that can be interpreted in its support – is essential to our theoretical approach. Roughly speaking, it means that the states of emotional feeling *carry meanings for the individual*. They *encode and exchange information* in interaction with other internal systems of representation, in a way essential to mathematical understanding and problem-solving performance.

For example, on one occasion the feeling of *frustration* may encode the failure of a recent series of trials of a particular strategy for solving a mathematical problem [“This isn’t working. . .”]. The frustration may evoke useful problem-solving heuristics leading to strategic change – such as trying a simpler, related problem. On another occasion, frustration may encode general information about the individual in the situation [“I can never think clearly during tests. . .”]. The feeling may rapidly give way to anxiety or despair, and evoke avoidance strategies. The meanings of our emotional feelings are highly context-dependent; far more, even, than the meanings of words and phrases. Often emotional meanings are unconscious or preconscious, and difficult to verbalize or interpret. These features, among others, make the empirical study of affect extremely difficult.

Viewing affect as representational suggests many parallels with cognition – e.g., in analogy with cognitive structures, we suggest the notion of affective structures, incorporating values, beliefs, attitudes, and associated pathways of emotional feeling (see below). Affect (as representation) also constitutes a system of communication *via* intonation, eye movements, facial expressions, ‘body language’, laughter, tears, noises, exclamations, etc. – a system at times highly ambiguous, yet working effectively in personal and social interactions.

2.2. *Affective pathways, competencies and structures*

Affective pathways are established sequences of (local) states of feeling that interact with cognitive configurations. Such pathways serve important functions for experts as well as novices – providing useful information, facilitating monitoring, and suggesting heuristic problem-solving strategies.

For example, an idealized positive pathway interacting with heuristics begins with *curiosity* and *puzzlement*. These feelings evoke exploratory and problem-defining heuristics, and motivate the solver to better understand the problem. *Bewilderment*, suggesting cognitive impasse and lack of insight (imagistic representation), leads to *frustration*. These feelings encode the information that up to this point, the strategies employed have led to insufficient progress. After strategic rethinking, *pleasure*, *elation* and *satisfaction* occur as the problem yields to new approaches. Global structures are built that entail positive self-concepts and anticipation of positive affect in association with difficult mathematics problems.

In contrast, a negative pathway also begins with curiosity and puzzlement, but these may encode a search for appropriate or safe procedures, rather than an exploratory opportunity. When procedures fail, the resulting frustration turns to *anxiety* and *despair*. These emotional feelings may evoke heuristic processes of a different sort – reliance on authority, defense mechanisms, avoidance and denial. Global structures of mathematics- and self-hatred are built.

Affect may thus empower or disempower students in relation to mathematics. Empowering affect serves as an impetus to persevere, take risks, engage with new external and internal representations, ask questions, or construct new heuristic plans. Disempowering affect hampers performance, blocks understanding or makes it unrecognizable when it occurs, and induces negative outcomes associated with mathematics anxiety or phobia. Thus it is meaningful to characterize *affective competencies* – the individual's capabilities that depend on appropriate affective encoding of strategically relevant information. Examples include the ability to act on curiosity, or to take frustration as a signal to modify strategy.

In our view, each person constructs complex networks of affective pathways and competencies, augmenting or diminishing mathematical problem-solving power, and carrying context-dependent meanings for the individual. They constitute part of what we here call *affective structures*. Among the important structures in relation to mathematics are mathematical intimacy and mathematical integrity, discussed below, as well as *mathematical self-identity*, discussed elsewhere (Lesh and Doerr, 2003; Goldin, in press).

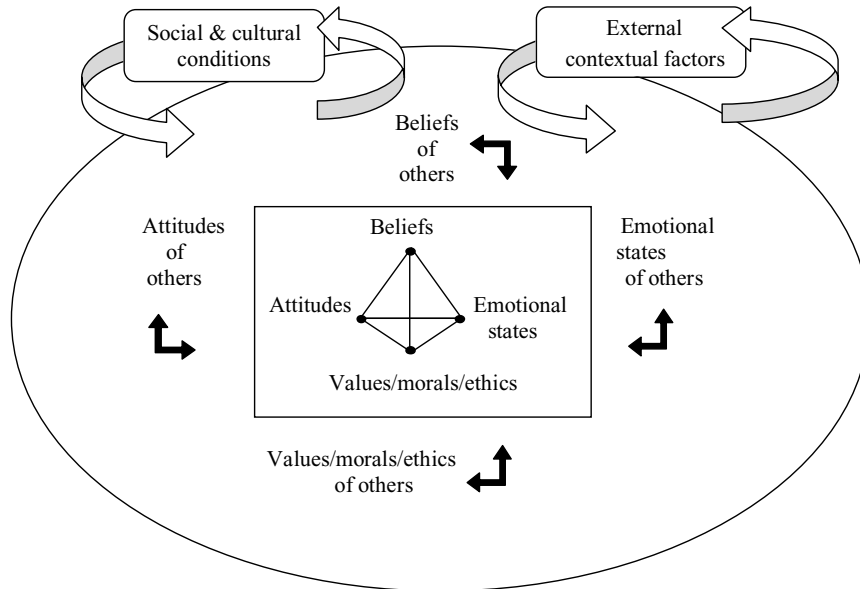


Figure 1. A tetrahedral model describing domains of affect.

2.3. Subdomains of affect: A tetrahedral model

McLeod and his collaborators partitioned the affective domain into three subdomains: *emotions*, *attitudes*, and *beliefs*. We extend these to a fourth subdomain of *values/morals/ethics*, creating a tetrahedral model (Figure 1). The descriptions that follow reflect our current view of these subdomains.

Emotions describe rapidly-changing states of feeling experienced consciously or occurring preconsciously or unconsciously during mathematical (or other) activity. Emotional feelings range from mild to intense, and are local and contextually-embedded.

Attitudes describe orientations or predispositions toward certain sets of emotional feelings (positive or negative) in particular (mathematical) contexts. This differs from the more common view of attitudes as predispositions toward certain patterns of behavior. Attitudes are moderately stable, involving a balance of interacting affect and cognition.

Beliefs involve the attribution of some sort of external truth or validity to systems of propositions or other cognitive configurations. Beliefs are often highly stable, highly cognitive, and highly structured – with affect interwoven in them, contributing to their stabilization.

Values, including *ethics* and *morals*, refer to the deep, ‘personal truths’ or commitments cherished by individuals. They help motivate long-term choices and shorter-term priorities. They may also be highly structured, forming value systems.

Each vertex of the tetrahedron in this picture may be understood as interacting dynamically with the others in a particular individual. Thus our emotional states influence and are influenced by our attitudes, beliefs, and values. One mechanism for this influence is the construction of global structures from recurring affective pathways, as described above.

In addition, each vertex interacts with the corresponding component in the affective domain of other individuals.

Finally, and very importantly, each individual's affect is deeply influenced by corresponding systems of the (mathematical or educational) sub-culture(s) wherein the person is situated. Thus the student's emotions, attitudes, beliefs, or values not only interact immediately with their counterparts in teachers or other students (in particular situations), but also interact over time with shared, normative emotional expectations, attitudes, beliefs, and values held by peers, school authorities, etc.

3. META-AFFECT

Meta-affect refers to affect about affect, affect about and within cognition about affect, and the individual's monitoring of affect through cognition (thinking about the direction of one's feelings) and/or further affect. We hypothesize meta-affect to be the most important aspect of affect (DeBellis, 1996; DeBellis and Goldin, 1997, 1999; Gomez-Chacon, 2000; Goldin, 2002). It is what enables people, in the right circumstances, to experience *fear* as *pleasurable* (e.g., in experiencing a terrifying roller coaster ride as fun), or to distinguish vicarious emotional feelings evoked by books or films from their 'real life' counterparts. Meta-affect helps guide the experience of *hypothetical* emotions, as these are used for cognitive gain.

Fear occurs frequently in mathematical contexts – e.g., when facing a mathematical problem, a particular mathematical topic, or a school examination. Advanced students or professional mathematicians may fear exposure of some perceived gap in knowledge or inadequacy in ability. The meta-affect associated with fear related to mathematics – i.e., the structure of feelings *about* the fear – is not usually pleasurable. But see below.

Meta-affect may ascend many levels. Such *towers of meta-affect* are usually powerful and stable – thus one may feel *guilt* about one's *anger* about the *pain* of perceived rejection for academic failure by a parent whom one *loves*. At the core, perhaps, is the love; but the negative meta-affect transforms it into something painful, and the anger and guilt contribute to an enduring, albeit dysfunctional, structure.

Careful consideration of meta-affect suggests to us that the most important affective goals in mathematics are *not* to eliminate frustration, remove

fear and anxiety, or make mathematical activity consistently easy and fun. Rather they are to develop meta-affect where the emotional feelings *about* the emotions associated with impasse or difficulty are *productive* of learning and accomplishment.

Cognition plays an important role in this meta-affect. Just as the knowledge that a roller coaster ride is ‘really safe’ can render fear pleasurable, mathematical exploration in an environment where the student knows making mistakes is ‘safe’ can transform negative emotions into positive ones. Beliefs, values, and ethics also play a role in meta-affect, as they influence how an emotion functions ecologically in the individual’s personality – i.e., the purpose served by the emotional feeling.

For example, frustration *could* and *should* indicate that a mathematical problem is nonroutine and interesting. It should carry with it anticipation of possible elation at understanding something new, or achieving a difficult goal. Then frustration itself is experienced as interesting, curious, even euphoric. Related, ‘cognitive’ beliefs and values in relation to mathematics can contribute to the construction of powerful meta-affect – e.g., the belief that eventual success is indeed likely, together with the intrinsic value placed on learning an advanced concept, or meeting a challenge. A supportive classroom culture provides a sense of *safety* in being ‘stuck’. Then frustration *coupled with productive meta-affect* suggests the problem is worth pursuing, and motivates further exploration rather than disengagement.

4. MATHEMATICAL INTIMACY

Mathematical intimacy involves deeply-rooted emotional engagement, vulnerability, and the building of mathematical meaning and purpose for the learner. It characterizes certain structures of emotional experience in powerful problem solvers, describing a possible intrapsychic relation between the self and one’s (internally represented) mathematics that connects with one’s internal sense of self-worth (DeBellis, 1998).

Intimate *behaviors* refer to observable verbal or nonverbal actions. Depending on the context, these may include the problem solver’s self-placement especially close to the physical work, cradling the paper with the arm or hand as if to say ‘this is mine,’ hesitation in sharing the work, closing the eyes as if to ‘feel’ the mathematics, breathing deeply, tending with great care, being so consumed by the interaction as to ignore environmental inputs (such as a teacher calling the student’s name), or speaking in an especially halting, quiet, or excited way.

Intimate mathematical *experiences* include emotional feelings of warmth, excitement, amusement, affection, sexuality, time suspension,

deep satisfaction, 'being special', love, or esthetic appreciation accompanying understanding. They may involve the person's internal representation of loved or respected ones, e.g., a sense that 'My father would be proud of me for this.' These experiences are more than merely enjoyable or otherwise positive; they *build a bond* between the personal knowledge constructed and the mathematical content. This may help explain why intimate learners often find it difficult to 'let go' of a misconception – the emotional dynamics associated with the affective structure of intimacy can encode a type of 'bonded knowledge' not easily separated from the self.

However, intimate engagement does not guarantee a positive long-term relationship with mathematics. A solver may feel disappointed, angry, or *betrayed in intimacy* by unexpected outcomes, failures, negative reactions from loved ones, rebuke from a trusted teacher, or scorn from peers. The pain of intimate betrayal is increased by the vulnerability associated with intimacy. Intimate betrayal does not distinguish between the mathematically talented and the mathematically challenged individual – it seems to be part of the human problem solving process (DeBellis and Goldin, 1999). Learning how to survive it intact is a meta-affective capability, possibly distinguishing individuals we would characterize as talented. This survival capacity, crucial to the person's mathematical development, may involve the components of mathematical integrity discussed next.

5. MATHEMATICAL INTEGRITY

We earlier used the term *mathematical self-acknowledgment* to describe a learner or problem solver's ability (or willingness) to acknowledge insufficiency of mathematical understanding (DeBellis, 1996; DeBellis and Goldin, 1997). Later we used the term *mathematical integrity* (DeBellis and Goldin, 1999) to reflect a more comprehensive interpretation.

Mathematical integrity describes an individual's affective psychological posture in relation to when mathematics is 'right,' when a problem solution is satisfactory, when the learner's understanding suffices, or when mathematical achievement deserves respect or commendation. Thus it involves, in relation to mathematics, the person's commitment to 'truth' and understanding, and, possibly, sense of moral character. It entails honesty and a degree of openness. We conjecture that students with strong mathematical integrity structures have the potential to engage in powerful learning and problem solving – especially if their mathematical integrity is interacting with their capability for mathematical intimacy.

Important components of this affective construct include: (1) recognition of an insufficiency of mathematical understanding or achievement,

(2) the decision to take further action, and (3) the nature of the action. In any particular mathematical situation, there may be no recognition of lack of understanding, partial recognition, or full awareness; whatever the circumstance, the individual may or may not take further action. If action is taken, mathematical performance may be helped or hindered depending on the action – e.g., looking for a deeper structure in a problem, solving a related problem, making a surface level adjustment (‘mathematical bluffing’), or stopping work.

Moreover, a problem solver may admit that something does not make mathematical sense only to himself or herself, or may also admit it to a teacher or another student. Either acknowledgment may pose specific value, moral, or ethical dilemmas. Vinner (1997) describes ‘pretending’ as a behavior of students trying to get credit when they “know [they] do not know,” and the moral dilemma posed by educational systems that reward rapid obedience to mathematical rules over understanding. When “an individual is not aware of the fact that he or she uses pseudo knowledge in order to get credit,” the issue of integrity becomes still more difficult. In our view, the student in this situation is not sufficiently *intimate* with the mathematics to recognize insufficiency in understanding.

How best to characterize individuals’ mathematical integrity structures, how consistent they are across problem situations, and how integrity is influenced by or develops within the social and cultural environment, are for us essential research questions for understanding mathematical learning and fostering students’ mathematical ability.

6. AN EMPIRICAL STUDY

6.1. *Methodology*

We conducted and videotaped a series of five clinical, task-based interviews with individual children, across two years. The overall goal was to infer in detail the development of individual children’s internal representations. Each interview (approximately 45–60 min) consists of several mathematical tasks, including at least one difficult, non-routine problem. The clinician (a mathematics educator) follows a detailed, rehearsed script designed by a research team. Present are two camera-crew members and one observer, as unobtrusive as possible. Details of interview scripts, concrete materials available to the students, and further elaboration of findings are described elsewhere (DeBellis, 1996; see also Goldin, 1998a; Goldin et al., 1993).

Of 22 children aged 9 to 10 years in four New Jersey public schools who began the study, 19 completed all five interviews. Four subjects, chosen to

differ in their observable levels of affect, were selected for longitudinal, in-depth affective analysis.

We distinguish consistently between *observation* and *inference*. The complexity of affect (from our representational perspective) suggests the importance of many ‘windows’ – i.e., distinct information sources, together with systematic methods of drawing inferences about internal affect. Our ‘windows’ included as primary data the videotaped interviews, the child’s productions, and the clinician’s and observer’s notes. We obtained or constructed: (1) general background information about the child; (2) complete, detailed interview transcripts, including speech, changes in tone of voice, timed pauses, descriptions of observable facial expressions, hand and body movements, posture, and schematic representations of original diagrams drawn or other external representations produced; (3) complete tabulation of affective verbal expressions, including both descriptive and non-standard, interjected affect-expressive words and sounds (examples below); (4) tabulation of instances of affect interacting with executive control, inferred from protocol analysis; (5) general impressions about the child’s affect from individual mathematics educators, not part of the research team, independently viewing the videotapes; (6) cognitive analysis with emphasis on affective interactions in a nonroutine problem; (7) results from an independently-developed affective coding scheme for facial expressions – the Maximally Discriminative Facial Movement Coding System (MAX) (Izard, 1983); (8) identified evidence of meta-affect; (9) identified evidence relating to mathematical integrity; and (10) inferences permitting description of affective pathways.

A research team designed protocol scripts, conducted pilot interviews and revised scripts, and carried out the main interviews. Transcripts report what was observable, audible, and inaudible. We consulted certified experts in MAX for inferring emotions from facial movements – here codes for three areas of the face are compared to predefined MAX codes for inferring emotion – e.g., a particular triple of codes at a particular time suggests ‘surprise’. Spoken words transcribed for each time frame were aligned with the corresponding MAX codes, permitting inferences about interactions between affect and cognition.

All four subjects utter sounds and affect-expressive words, specific to the individual. Sounds include: ‘uhhh, ummm, hmmm, and/or ummm’ (when searching for a strategy), ‘errr’ (a low rumbling growl, associated with apparent confusion or frustration), ‘euwww’ (struggling to solve a non-routine problem), sighs (after finding a strategy failed), or ‘oohhh’ (accompanying a new idea). Affect-expressive words include: ‘like’, ‘love’, ‘fun’, ‘happy’, ‘confused’, ‘challenging’, ‘hard’, ‘worried’, and ‘scared’ (DeBellis, 1996). A complete tabulation for each subject shows

the type and frequency of affective expressions varying from individual to individual.

As this partial account suggests, we infer affective states from a variety of observations, repeatedly viewing tapes, reflecting, discussing and analyzing. Nevertheless, our methods have serious scientific limitations and should be viewed as purely exploratory. For example, we have no independent way to verify our inferences about the children's emotional feelings, or to compare systematically inferences by different observers.

Our empirical study, described elsewhere in considerable detail (DeBellis, 1996), provides numerous examples illustrative of our theoretical constructs. Here we shall include just a few. Our affective analysis of task-based interview protocols helped us to formulate the conjectures and hypotheses surveyed in the discussions of Sections 1–5.

6.2. *An example pertaining to meta-affect*

One important feature of meta-affect is that it allows the solver to experience *hypothetical* emotion to help inform cognition.

Early in Interview #3, we infer that 'Londa', age 10, feels *discomfort* or *unease* at the unfamiliarity of the interview situation (three videocameras, microphones, two camera crew members, a tall male clinician, a female observer), coupled with nervous anticipation. We infer this from her facial expressions and body movements. In a sense, her emotion is 'real time' and straightforward.

Later in the interview, Londa is asked "Which would be easier, to cut a birthday cake into three equal pieces or four equal pieces?" Three Styrofoam objects are on the table; a circle, a rectangle and an equilateral triangle. She replies, "Like if you have a circle and cut it in three's that would be really hard because it would be something like that, like that [motions with finger] or if, if you did it like this [repeats the same gesture indicating the cutting] and you found out they weren't equal, you would have a hard time deciding. But if you had this [reaches for the rectangle] all you'd have to do is cut it here and here."

From gestures, facial expressions, and body movements, we infer that Londa is envisioning/imagining what cutting the circle in thirds would be like, a process that includes letting herself feel *discomfort* to encode how difficult it would be to decide where to cut. She *allows* herself to feel the emotion, in a hypothetical context. Unlike her discomfort at the beginning of the interview, this is an emotion she controls and feels entirely *comfortable* about feeling. It is not merely milder. She has shaped the context herself; the meta-affect is positive and safe.

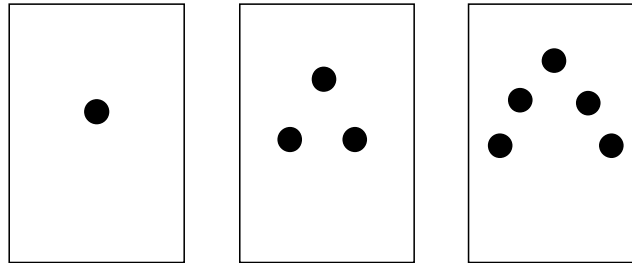


Figure 2. The first three cards in a sequence with a chevron pattern.

Although we use the same word, ‘discomfort’, to describe the two situations, an empathic observer notes in the second case some mismatch between the observed behavior and the emotion inferred. When Londa initializes and controls the emotion, she experiences it in a ‘virtual’ way, using it to achieve some particular cognitive gain. As this example illustrates, to say a problem solver feels any particular emotion can be a tremendous oversimplification.

6.3. *An example pertaining to mathematical integrity*

‘Jacqueline’ is in 4th grade, age 9, solving a problem during Interview #1. The clinician places three cards (see Figure 2) in front of the student and, in slow succession, asks the following questions, “What do you think would be on the next card? Why? Can you show me what you mean? Do you think this pattern keeps going? How would you figure out what the 10th card would look like?” After a complete, coherent response has been elicited, the child is asked to find how many dots would be on the 50th card in this card sequence. Jacqueline appears intimately engaged, but has not created the geometrical chevron pattern when asked to ‘show’ what she meant.

Clinician: “How many dots do you think would be on the 50th card?”

Jacqueline: [opens eyes wide, raises eyebrows] “Fifty?”

Clinician: “A huh, 50th card?”

Jacqueline: [6 second pause, sits back in the chair, arches her back, opens eyes wide, raises eyebrows, presses lips together] “I think we’re gonna have to multiply”. . . “because we can’t write 50 cards.” [raises eyebrows]. . . [smiles] “because that’s too much. And you can’t do this all the time. Sometimes you got to multiply to get finished easier.” She decides to multiply 19×5 (19 dots on the 10th card). . . [11 second pause] “This is what I’m not good at. [smiles]. . . Think that’s right?”

Clinician: “What do you think?”

Jacqueline: “I don’t think so. . . Maybe if we tried um. . . maybe divide it” [looks up] “no. . . this is 19” [points to the 10th row of her figure].

Jacqueline acknowledges insufficiency of understanding, makes a change, proposes a new strategy, and tries to use it, a total of 10 different times – displaying high perseverance. Observed aspects of mathematical integrity include her identifying errors in her thinking and computation, and verbalizing a strong desire to ‘get the problem right’. This establishes an affective posture allowing her to continue working, even when making little mathematical progress.

While Jacqueline shows some signs of intimate engagement, she does not participate fully, falling short of discovering the underlying pattern. Ultimately, she is persuaded by an incorrect solution.

6.4. *An example pertaining to mathematical intimacy*

‘Jerome’ is in 5th grade, age 10, solving a problem with jelly beans during Interview #3. Two glass jars, one with 100 green and one with 100 orange jelly beans were placed on the table. The clinician asks, “Suppose you take 10 green jelly beans from the green jar and put them into the orange jar and mix them up. Then suppose you take 10 jelly beans from the mixture and put them back into the green jar. Which jar would have more of the other color jelly beans in it?” At a certain point Jerome inhales deeply, brushes back his hair, pauses 4 seconds, raises his eyebrows, and smiles: “I don’t know [15 second pause, stares at jelly bean containers, raises his eyebrows, shakes his head from side to side, sits back in the chair].”

Clinician: “What do you think is going on?”

Jerome: “I dunno. . .”

Clinician: “What are thinking about?”

Jerome: “Uhhh. . . I’m just trying to figure out how did this happen. [17 second pause, raises eyebrows, furrows his eyebrows, presses his lips, looks upward, shakes his head from side to side] I dunno.”

Jerome’s behaviors, from which we infer his *intimate engagement*, include close proximity to the jelly beans when experimenting, cradling his work as if to separate his activity from the clinician, raising his voice, breathing deeply, brushing his hand through his hair, shrugging his shoulders, facial expressions (raised eyebrows, furrowed brow, pressed lips, wide eyes, and smiles) and silent pauses. He sits back in his chair as though

distancing himself when the outcome contradicts his expectation – a *protective* movement, suggesting feelings of vulnerability.

6.5. *Comments on the data for 'Frank'*

The data for 'Frank', (Op't Eynde and Hannula, this issue), include a brief overview about him, a transcribed problem-solving episode that includes verbalizations together with observational comments regarding facial and body movements drawn from a videotape of his classroom work, and student interviewer comments during a video-based stimulated recall interview (VBSRI).

The VBSRI provides an interesting, additional 'window' on affect, different from any in our study. One can compare affective inferences drawn from actual-time data with the individual's conscious, retrospective self-descriptions. A discrepancy between these (which, apparently, did not occur with Frank) might then suggest interesting meta-affect – the student's feelings about his feelings being such as to influence him to transform them retrospectively.

Based on the available data, we cannot draw independent inferences with confidence regarding Frank's affect. Emotionally expressive words (“... then I panic...”, “... you don't feel well...”) occurring during the VBSRI, not the original problem-solving episode, could be taken at face value. From Frank's facial expressions (“brows drawn together, slightly lowered”) the investigators infer worry (rather than, for example, curiosity or determination), possibly using other visual cues. From his statement, “Come on, what is this all about!” they infer anger. The latter inference presumably makes use of Frank's tone of voice, as the words alone could be consistent with enhanced curiosity, determination, impatience, or frustration. Through our study we learned that interpreting affect is very tricky; thus we advocate using a multiplicity of data sources and analysis techniques, aiming for consistent inference methods open to discussion and reexamination by others. To avoid premature or highly subjective inferences, one also needs considerable reanalysis while reviewing the videotapes. However, our inability to arrive at independent inferences does not necessarily call into question the actual inferences made by Op't Eynde, DeCorte, and Verschaffel (this issue) about Frank, using additional sources of data such as questionnaires, videotapes, and results of systematic coding systems including MAX, and with opportunities for reflection and analysis.

From our theoretical standpoint, Frank's inferred affect – especially the sequence in his 'affective profile' given as “worried, frustration/fear/anger, happy” (Op't Eynde, DeCorte, and Verschaffel, this issue) – describes an 'affective pathway'. This pathway combines some features of the idealized

pathways described in Section 2; this portion of it seems closest to the first, ‘positive’ pathway. To apply our further theoretical notions to Frank, we would need an analysis of interactions between emotional states and cognition, a discussion of the meanings encoded in Frank’s emotional feelings (from a representational perspective), and some consideration of the meta-affect surrounding those feelings – e.g., to what extent is Frank’s ‘anger’ controlled or modified by other emotions? Does he possibly feel ‘pride’ in his anger, signifying his assertion of a self-image of usually being able to solve mathematical problems without the calculator?

7. CONCLUSIONS

We have sought to infer and characterize aspects of affect within individual children, in the context of their solving mathematical problems. The occurrence in learners of well-defined pathways of local affect has implications for the development of an individual’s global attitudes and beliefs toward mathematics, and conversely. Furthermore affect has meaning; and one must attend to meta-affect in order to understand the affect and its meaning. Finally the development of powerful affective and meta-affective structures, like those of *mathematical intimacy* and *mathematical integrity*, may turn out to be keys that unlock mathematical power in learners.

Our framework provides a structure that accommodates many of the ideas presented in this special issue. While we developed it originally to help understand internal representation, especially interactions between affect and cognition during problem solving, it is quite resonant with Malmivuori’s work on the dynamic aspects of personal learning as connected to students’ self-system processes, and with Hannula’s work on motivation and its potential to be self-regulated. These two articles describe aspects of the affective system internal to the individual. Hannula, as well as Brown and Reid, present ideas about how the affective system interacts with planning and executive control. Hannula discusses aspects of motivation and goals, while Brown and Reid explore how emotion may interact with decision-making.

The tetrahedral model we describe situates internal aspects of affect (inside the rectangle of Figure 1) within an affective environment (outside the rectangle). As the individual interacts with other individuals, so too do their affective structures interact, influenced by external contextual factors and social and cultural conditions. The articles by Op’t Eynde, DeCorte and Verschaffel and by Evans, Morgan, and Tsatsaroni begin to analyze affect in this complex environmental system. The first group’s work focuses on describing the dynamics through a socio-constructivist perspective, while

the latter work describes how emotions may be socially organized through discourse and power relations. The lenses and foci are quite different from ours, yet we see the potential for unification.

We think a research agenda on affect should move toward investigating these ideas within a more unified framework, so that our research can valuably inform the mathematics education community.

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