CHAPTER 4

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AFFECT, META-AFFECT, AND MATHEMATICAL BELIEF STRUCTURES [1]

Abstract. Beliefs are defined here to be multiply-encoded, internal cognitive/affective configurations, to which the holder attributes truth value of some kind (e.g., empirical truth, validity, or applicability). This chapter offers some theoretical perspectives on mathematical beliefs drawn from analysis of the affective domain, especially the interplay between meta-affect and belief structures in sustaining each other in the individual.

1. INTRODUCTION

Research in mathematics education has tended to focus principally on cognition, and far less on affect. This may be due, in part, to the popular myth that mathematics is a purely intellectual endeavor in which emotion plays no essential role. Valerie DeBellis and I, in developing a language for careful discussion of the affective domain in mathematics, seem to have introduced several rather uncommon ideas. This chapter offers some theoretical perspectives on mathematical beliefs, drawn from analysis of the affective domain, based on some of those ideas. My main assertion is that the stability of beliefs in individuals has much to do with the interaction of belief structures not only with affect (feelings), but with meta-affect (feelings about feelings) - that through their psychological interplay, meta-affect and belief structures sustain each other.

The chapter is organized as follows. First I mention several important perspectives on affect as a system of representation encoding information, intertwined with cognitive representational systems, and as a language for communication having an important cultural dimension. Next I consider the key construct of meta-affect, including affect about affect, affect about and within cognition that may again be about affect, monitoring of affect, and affect as monitoring. We shall see that powerful affective representation inheres not so much in the affect as in the meta-affect. A basic idea here is that affect stabilizes beliefs, and beliefs establish meta-affective contexts.

I then turn to a discussion of beliefs, belief structures, and belief systems. Beliefs are defined to be multiply-encoded cognitive/affective configurations, to which the holder attributes some kind of truth value (e.g., empirical truth, validity, or applicability). I distinguish among: working assumptions or conjectures; weakly- or

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strongly-held beliefs; individual and shared beliefs; belief structures and systems of belief; warrants for beliefs; psychological functions of beliefs and belief systems; knowledge (beliefs that in some sense *apart* from the fact of belief or the fact of warranted belief, are true or valid); and individual and shared values and value systems. Then a preliminary typology of mathematically-related beliefs is offered, organized not by who holds them but by their content domain. Belief structures can intersect several of these categories.

Finally I consider how belief structures, warrants for belief, and meta-affect can establish and sustain each other - at the social level, as well as in the individual - and explore some warrants for mathematical beliefs in relation to affective structures. Changes in mathematical belief structures require, and entail, changes in affect as well as cognition.

2. SOME PERSPECTIVES ON AFFECT

Let me begin by referring briefly to a few of the perspectives taken here regarding affect. For related and sometimes contrasting perspectives, see Leder (1982,1993), McLeod (1988, 1989, 1992), McLeod and Adams (1989), Drodge and Reid (2000), and Gomez-Chacon (2000). A useful overview of research on affect in mathematics education is given in McLeod (1994).

First, affect is seen as one of several internal systems of representation in individuals (cf. Zajonc, 1980; Rogers, 1983; Goldin, 1987, 1988, 1998, 2000; Picard, 1997). That is, the affective system does not merely accompany cognition, or occur as an inessential response to cognitive representation, but affect itself has a *representational* function. Affect meaningfully encodes information. This includes information about the external physical and social environment (e.g., feelings of fear encoding danger), information about the cognitive and affective configurations of the individual herself or himself (e.g., feelings of bewilderment encoding insufficiency of understanding, feelings of boredom encoding absence of engagement, or feelings of loneliness encoding absence of intimacy), and information about the cognitive and affective configurations of others, including social and cultural expectations, as represented in and projected by the individual (e.g., feelings of pride encoding satisfaction taken by one's parents or teachers in one's achievements).

When individuals are doing mathematics, the affective system is not merely auxiliary to cognition - it is central. However affect as a representational system is *intertwined with cognitive representation.* Affective configurations can stand for, evoke, enhance or subdue, and otherwise interact with cognitive configurations in highly context-dependent ways. The very metaphors used in thinking may carry positive or negative affect.

Systems of cognitive representation, specifically the verbal/syntactic, imagistic, formal notational, and strategic/heuristic systems that I have discussed extensively elsewhere in formulating my developing model of mathematical problem-solving competence, function in part by evoking affect and the representational information that it encodes. In earlier work DeBellis and I focused particularly on the interplay between affective states and the heuristic or strategic decisions taken by students during problem solving (DeBellis & Goldin, 1991, 1993). For example, feelings of frustration while doing mathematics may encode (i.e., represent) the fact that a certain strategy has led down a succession of "blind alleys", and (ideally), these feelings might evoke a change of approach.

Emotions are biologically based, and there is evidence for emotional systems having evolved in other species. Affect does much more than inform and motivate individuals. It serves also as an extraordinarily powerful *evolutionary language for communication,* that is essentially human. Each individual person's affect interacts with that of other people (often quite tacitly or unconsciously, often very powerfully), so that crucial information is exchanged. The specifics of this communicative system, which functions through "body language", eye contact, facial expressions, tone of voice, and scent, as well as spoken language, cries, laughter, and other noises and interjections, seem to have evolved alongside the emergence of the human species. The sharing of affect among pairs or groups of people is generally *essential to human survival.* In discussing the influence of affect on beliefs, then, it is important to take note of, and distinguish between, individual affect and shared affect. Progress is being made in understanding the neuroscientific basis of affect, which allows informed discussions of the role it plays in our conscious awareness (cf. Damasio, 1999).

In the individual, we can distinguish certain subdomains of affective representation (McLeod, 1988, 1989; DeBellis, 1996; DeBellis & Goldin, 1997): (1) *emotions* (rapidly changing states of feeling, mild to very intense, that are usually local or embedded in context), (2) *attitudes* (moderately stable predispositions toward ways of feeling in classes of situations, involving a balance of affect and cognition), (3) *beliefs* (internal representations to which the holder attributes truth, validity, or applicability, usually stable and highly cognitive, may be highly structured), and (4) *values, ethics, and morals* (deeply-held preferences, possibly characterized as "personal truths", stable, highly affective as well as cognitive, may also be highly structured).

Likewise, shared affect refers not only to transient, shared emotions (e.g., intimate excitement between lovers, pleasant laughter among a group of friends, tension in a mathematics class before an examination, or swelling group enthusiasm during a well-led problem solving discussion), but also to complex, shared, and possibly very powerful structures of feeling that are culturally embedded (e.g., religious reverence, or nationalist fervency), involving attitudes, beliefs, and values.

It is important to be able to discuss affective competencies and affective structures, in a way somewhat analogous to discussions of cognitive competencies and structures. Among the important constructs here are the distinction between local (transient, special-context) and global (long-term, multi-context) affect; the notion of affective pathways and networks (recurring sequences and links among emotional states), with accompanying meanings to the affective configurations; defense mechanisms (affective structures that serve to protect the individual from experiences of emotional hurt or pain); and processes of change in global affect (e.g., passage by an individual from long-held anger to forgiveness).

Constructs especially important to the doing of mathematics include the notions of *mathematical intimacy,* involving vulnerability, personal caring, private experience, and possibly creative expression; and *mathematical integrity,* including, e.g, self-acknowledgment of inadequate understanding (DeBellis & Goldin, 1997, 1999; DeBellis, 1998; Vinner, 1997). A further, fundamental construct that I think has received insufficient attention is that of *mathematical self-identity.* By this I mean the spectrum of related affect and cognition, growing as the child learns and grows, that eventually may take the form of answers to the question, "Who am I?" in relation to mathematics.

3. AFFECT AND META-AFFECT

A central notion that DeBellis and I are developing is that of *meta-affect.* We introduced this term (DeBellis, 1996; DeBellis & Goldin, 1997) to refer to affect about affect, affect about and within cognition that may again be about affect, the monitoring of affect, and affect itself as monitoring. In many situations, the meta-affect is actually the most important aspect of the affect.

Consider, for example, the emotion of fear. One thinks first of fear as a negative state of feeling, signaling danger. In the absence of actual danger in the environment, fear might be seen as a counterproductive state, an incorrect encoding, a feeling to be avoided or soothed. A young child may be terrified of the dark, or of being alone. An adolescent may experience fear of rejection or of failure. Some people are terribly and involuntarily afraid of crowds, of heights, of flying in airplanes, or of public speaking. Some, of course, fear mathematics. In these situations, our first impulse is to try to assuage the feeling.

But a moment's reflection reminds us that in the right circumstances, individuals can find fear highly pleasurable. People flock to horror movies. They enjoy amusement park rides, where the more terrifying the roller coaster experience, the more exhilarating and "fun" it is. Why is this? The cognition that the person is "really safe" on the roller coaster *permits the fear to occur in a meta-affective context of excitement and joy.* The more afraid the rider feels, the more wonderful she feels *about* her fear. She may experience a satisfying sense of her own bravery, of having conquered fear, and the anticipatory joy (Vorfreude) of stepping onto the solid earth again to be regarded with admiration by friends.

Imagine, however, that a cable breaks during such a ride, and the roller coaster swerves uncontrollably. The experience changes entirely! Now the rider is "truly" afraid, as the danger is (believed to be) actual. This fear feels entirely different, because the *meta-affect* has changed. Even if the person is really in no danger, the removal of the *belief* that she is safe changes the nature of the affective state - no longer does it feel wonderful to feel so afraid. The terrifying ride, not fun any more, has become horrible.

What makes the different meta-affect possible? It might seem that the "cognitive" belief, that the ride is in fact safe, is the main essential to the joyful meta-affect. In this sense, the belief stabilizes the meta-affect. But a straightforward influence of cognitively-based belief on meta-affect cannot be the whole story.

Other beliefs and values (tacit or overt) play key roles, influencing the *ecological function* of the fear in the individual's personality - values of life and safety, of approval by peers or authority, of personality traits such as bravery. Yet even conscious cognitions, beliefs, and values do not suffice to account for all the meta-affect: an adult having a "panic attack" in a crowd or in an airplane may well "know intellectually" he is "really safe", not *want* to feel fear, but experience it involuntarily. *Unconscious defense mechanisms* prevent the person from "really" believing in the fact of his safety. Here, the meta-affect stabilizes some level of belief in the actuality of the danger.

Fear of mathematics - or even fear of a particular topic in mathematics, such as fractions, algebra, or word problems - is a common phenomenon. One student may experience fear immediately on being given a mathematical problem to solve; another upon realizing that he does not know how to proceed with the problem. Some may be afraid of the test, of the teacher, of the computer, of failure, or of disapproval at home. This fear may feel quite involuntary. The knowledge that one has studied hard and is well-prepared may or may not remove the feeling of fear, or embed it in positive meta-affect. Even - or perhaps I should say, *especially* - advanced graduate students of mathematics may fear exposure of (self-perceived) mathematical inadequacies.

The meta-affect of fear in doing mathematics is not usually joyful, though occasionally it can be. For instance, a bright high school student might be fearfully nervous before an interscholastic mathematics contest, with positive meta-affect - the "contained fear" can enhance the experience, as she already anticipates being able to say, "I was really nervous, really afraid ... I'm always that way at these contests ... and I did great!"

When we consider less extreme feelings, such as frustration during mathematical problem solving, there is a wide range of commonly occurring meta-affect. For some, the frustration signals anticipation of failure, with attendant negative emotions, so that the meta-affective context is one of anxiety or fear. But for another student solving the same problem, the experience of frustration may involve meta-affect that is positive. The student anticipates success, or at least a satisfying learning experience. The local affect of frustration signals (i.e., represents) that the problem is nontrivial, deep, or interesting, and heightens the anticipation of joy in success. The student's "cognitive" belief in her high likelihood of success, her confidence that mathematics yields to insightful processes, along with the high personal value she places on meeting challenges, may contribute to her feeling quite positive *about* the frustration - a very different meta-affective context.

Powerful affective representation that fosters mathematical success inheres not so much in the surface-level affect, as it does in the meta-affect.

4. BELIEFS, BELIEF SYSTEMS, KNOWLEDGE, AND VALUES

Furinghetti and Pehkonen (chapter 3, this volume) discuss alternative definitions and interpretations of *beliefs* and *knowledge.* Törner (chapter 5, this volume) stresses the absence of consensus, and proceeds to elaborate considerably on the notion of beliefs. Here I propose to define *beliefs* as multiply-encoded cognitive/affective configurations, usually including (but not limited to) prepositional encoding, to which the holder attributes some kind of *truth value*. The latter term is not taken in the technical sense of symbolic logic, but as a term that may variously refer to logical truth, empirical truth, validity, applicability to some degree of approximation, metaphysical truth, religious truth, practical truth, or conventional truth. Within mathematics, the "truth value" might be logical truth in the sense of deducibility from specific formal assumptions, or it might sometimes be conventional truth in the sense of satisfying some agreed-upon, arbitrary rules of definition or notation.

A *belief structure* is a set of mutually consistent, mutually reinforcing, or mutually supportive beliefs and warrants (see below) in the individual, mainly cognitive but often incorporating supportive affect. A *belief system* is an elaborate or extensive belief structure that is socially or culturally shared. Since I may be employing these terms rather differently from their casual uses, let me stress that my intent is to distinguish *individuals'* belief structures from *socially or culturally shared* belief systems.

A belief is, to begin with, individually held: for example, I believe that guiding children to discover logical patterns for themselves generally fosters their enjoyment and learning of mathematics. Such a belief may or may not be shared - some may agree with me, others may not. Belief structures, like cognitive structures, refer here to the *individual's* complex, personal, internal representational configurations: my belief about guiding children to discover patterns for themselves does not stand in isolation in my head; it is part of a structure of mutually reinforcing beliefs that I hold, together with a variety of reasons - or warrants - I have for holding them. Beliefs and belief structures are important in understanding individuals' mathematical problem solving heuristics and strategies (Schoenfeld, 1985; Lester, Garofalo, & Lambdin Kroll, 1989).

Belief systems, on the other hand, refer to socially or culturally shared belief structures, that are sufficiently broad to warrant the term. Shared beliefs, or belief systems are, in turn, not exactly the same as *normative* beliefs. The latter are idealized, approximate descriptions, at the societal or cultural level, of beliefs that one "should" hold (but may or may not *actually* be held by very many people). Relations among social norms and shared beliefs, the beliefs of individuals, and emotions are discussed further by Cobb, Yackel, & Wood (1989) and Cobb & Yackel(1996).

Note that it is the attribution of *truth* (of some kind) that turns mere propositions, conjectures, stories, or hypotheses, into beliefs. This attribution is by the holder, and not necessarily by others. It is not to be assumed, even when a belief is shared and normative, that the believer or believers are correct in their attributions of truth or validity to it. That is, some beliefs may - in fact - be false ones.

There is an unfortunate tendency among cultural relativists to use the word "knowledge" as if it were synonymous with "belief", "shared belief", "normative belief", "warranted belief", or some combination of these. For example, according to Confrey (2000), constructivism - which, in its radical formulation, has significantly influenced quite a few mathematics education researchers - holds as one of its "key concepts" the tenet that "knowledge is justified belief":

> In posing this claim I am not requiring that all knowledge is justified, but rather that if and when challenged, it can be justified. Insisting on only potential challenge is necessary so that in a stable body of knowledge, I can claim such statements as 3+7=10 as knowledge without requiring that I have actually justified it at the time. However, should someone ask me, how do I know that, I am obliged to produce a trajectory of acceptable reasoning and argument. If this does not result in convincing my audience, then the statement's status as knowledge is in question. Its validity will remain in doubt until an appeal to a larger and/or more qualified group of experts can be made successfully or until the previous challenge is resolved.
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> ... That is, constructivism entails a rejection of assured transcendent truth in our knowledge. (Confrey, 2000, p. 12)

Using the term "knowledge" this way leaves no convenient word to distinguish beliefs that are *in fact* true, correct, good approximations, valid, insightful, rational, or veridical, from those that are *in fact* false, incorrect, poor approximations, invalid, mistaken, irrational, or illusionary. I do not propose here to address the profound philosophical issues glossed over by the many different forms that relativism has taken. However, rejecting on first principles the very possibility of any such "in fact" distinction among competing, well-justified beliefs is in my view ultimately destructive of reasoned discourse - in mathematics and the empirical sciences, precisely because this distinction is essential to the integrity of the subject. The philosophical problems associated with "truth" do not disappear by rejecting it from the start as a requisite characteristic of knowledge. I have intentionally suggested a variety of possible interpretations, of which most do not require the "truth in our knowledge" to be either assured or transcendent.

It is important to characterize some related notions, that differ from or elaborate on the notion of belief. A *working assumption,* or a *conjecture,* is an internal configuration, often propositionally encoded, taken as a basis for exploration or discussion but (at least temporarily or provisionally) *without* attribution of truth, validity, or applicability. A *hypothesis* is such a proposition put forth with the explicit intent of investigating its possible truth or falsity (e.g., by gathering empirical data that tend to confirm or disconfirm it). A *viable* conjecture, hypothesis, or belief is one that has possibly proven useful or empowering, and has not to this point been disconfirmed in the judgment of the person or people making or holding it.

Viability is *not* the same thing as validity. A belief in the uncanny accuracy of astrological forecasting is personally viable for many people, and socially viable today in a large subculture (very likely, larger than the professional scientific subculture). But it is not valid.

A belief may be *weakly- or strongly-held* according to two different characteristics: the magnitude of the *importance* that is attributed to its being true (i.e., the believer may have a lot at stake), and the degree of *certainty* with which its truth value is attributed (i.e., the believer may have very convincing evidence).

Warrants for a belief consist in the believer's reasons or justifications for attributing truth, validity, or applicability to it. (Thus, in my language, Confrey's discussion pertains entirely to warrants for belief.) Warrants, like the beliefs that they warrant, may be personal or shared. In ordinary life, personal warrants (whether or not shared) include direct observations, indirect reports, plausible inferences from observations and reports, rational deduction from or compatibility with other beliefs, assertions of authority, etc. Some personal warrants may never be intended to convince anyone else (e.g., "My parents raised me to believe it.") In mathematics, the usual shared warrants include verification and proof making use of agreed-upon reasoning processes. In the natural sciences, they include goodness of fit with observation and measurement, compatibility with the outcomes of controlled and repeatable experiments, theoretical coherence, and parsimony. Depending on their nature, some warrants may be stronger than others - one who attributes validity to a scientific hypothesis after it has been repeatedly verified through controlled experimentation normally has a stronger case than another who attributes validity based on informal observations and anecodotal reports.

Many beliefs have *psychological functions* in the believer, and these are most often of an affective nature. A saying attributed to the U.S. journalist Henry Louis Mencken (1880-1956), "People will believe what they want to believe", has considerable descriptive validity. A belief or belief system may contribute essentially to the holder's self-identity, to the coherence of the believer's world view, or to the sense of certainty in his or her values. Thus it is important to be able to speak of the affective consequences of beliefs, and the affective contexts in which beliefs are held. This applies to all of us. If my belief structure that includes the invalid nature of astrology were to crumble, some of my self-identity as a scientist might be called into question. Of course, this provides not the slightest *scientific* warrant for my holding the belief, but it may help account psychologically for the importance I attach to it.

Knowledge (in the technical sense of this article) refers to beliefs that, in a sense *apart* from the fact of belief or the acceptance of warrants for belief by an individual or group, are true, correct, valid, veridical, good approximations, or applicable. Sometimes the term knowledge may be restricted further, to warranted or even well-warranted beliefs that have one or more of these characteristics.

Beliefs should also be distinguished from *values,* with which they are often closely entwined. The distinction can be subtle, because the latter are sometimes called beliefs - in ordinary speech, I might say that "I *value* learning", or (more or less equivalently) that "I *believe* learning is valuable". The desired distinction is psychological, not philosophical - values have to do with what is held to be good, worthy, or desirable (rather than with what is held to be logically or empirically true), and are thus fundamentally matters of personal choice. Of course, when an individual further accords "truth" to a statement of value, seeing it as validated by religion, authority, or social consensus, it becomes a belief as well as a value; but this does not always happen.

A complex, variegated system of shared values, morals, or ethics and expectations is a component of cultural representation. Developed in the individual from childhood (Kohlberg, Levine, & Hewer, 1983), values/ethics/morals comprise a component of the affective system of internal representation in the individual, and one of the most powerful motivators of human beings - driving us to define our life purposes, to feel right or wrong, justified or guilty, to judge others as right or wrong, and to engage in creative, altruistic, or extraordinarily destructive behavior.

The system of values/ethics/morals usually influences beliefs, and provides a partial (or, sometimes, total) basis for them. In doing mathematics, for instance:

> Following the rules, or following directions (including mathematical rules), may be regarded by the child as 'good', failing to do so as 'bad'. To us this is much more than a belief about what mathematics is, or what works to obtain solutions. Some students who do not follow established instructional procedures, as in addressing a non-routine problem, may actually be tacitly contravening their own moral values or self-expectations, while others (who value originality, rebellion, or self-assertiveness) may be acting consonant with them. Cheating in school may be considered evil or shameful, and doing mathematics with help may for the child be a form of cheating. The tacit commitments made by students to learn and to understand, their sense of goodness about themselves when they do as they 'should' do, and wrongness when they fail to do as they 'should', all fall within this component. (Goldin & DeBellis, 1997, p. 212)

Now, very different notions of truth or validity pertain in various contexts to beliefs. I have not been able to discover who first offered the observation, "The belief that something is so does not make it so", an assertion generally regarded as essential to scientific thinking. Nevertheless, this holds in some but not all contexts. With regard to the physical world, or with regard to mathematical questions *after* conventions have been established and axioms accepted, the mere fact that something is believed is *not* a valid warrant for it, and does not *per se* influence its truth. In some psychological contexts, a belief may have a *partial* influence on its own truth - for instance, with regard to an individual's estimation of his or her own mathematical ability. In still other contexts, where beliefs are based wholly on personal or shared values, beliefs can *create their own truth* in a self-referential manner – e.g., belief in one's own courage, in the beauty of the beloved, in the value of honesty, or in the elegance of the proof of a theorem.

5. SOME TYPES OF MATHEMATICAL BELIEFS

This preliminary typology of the kinds of beliefs of interest to mathematics educators is organized not by which individuals, groups, populations, or cultures might hold them, but by the nature of their content domain. It is included in order to lend specificity to the general points I have made. Op 't Eynde, De Corte, and Verschaffel (chapter 2, this volume) discuss similar categorizations by Underhill, McLeod, and others, offering a comprehensive overview. Belief structures in the individual, and belief systems occurring in social groups, often intersect several of the categories.

- Beliefs about the physical world, and about the correspondence of mathematics to the physical world (e.g., number, measurement);
- Specific beliefs, including misconceptions, about mathematical facts, rules, equations, theorems, etc. (e.g., the law of exponents, the quadratic formula, the idea that "multiplication always makes larger");
- Beliefs about mathematical validity, or how mathematical truths are established;

- Beliefs about effective mathematical reasoning methods and strategies or heuristics;
- Beliefs about the nature of mathematics, including the foundations, metaphysics, or philosophy of mathematics;
- Beliefs about mathematics as a social phenomenon;
- Beliefs about aesthetics, beauty, meaningfulness, or power in mathematics;
- Beliefs about individual people who do mathematics, or famous mathematicians, their traits and characteristics;
- Beliefs about mathematical ability, how it manifests itself or can be assessed;
- Beliefs about the learning of mathematics, the teaching of mathematics, and the psychology of doing mathematics;
- Beliefs about oneself in relation to mathematics, including one's ability, emotions, history, integrity, motivations, self-concept, stature in the eyes of others, etc.

For some of these kinds of belief, there exist fairly well-defined, culturally normative systems within the mathematical community. For others, the norms vary or scarcely exist. In all cases, there is the question of the interface between the individual and the social: How does the individual's affective representational system interact with those around her - as the belief or belief structure first forms, and as it evolves in the doing of mathematics?

6. INTERFACES BETWEEN THE INDIVIDUAL AND THE SOCIAL

We have *within* the individual (1) personal emotions, (2) personal attitudes, (3) personal belief structures, and (4) personal values/ethics/morals. Likewise, in the individual, there is the capacity to represent each of these in *other* individuals, especially when one person may be the object of another's intense feeling. Furthermore individuals have the capacity to represent the notion of *normative* or *appropriate* emotions, attitudes, beliefs, and values/ethics/morals, to evaluate their own feelings in relation to such norms, and to experience accompanying feelings about their feelings - guilt at having an inappropriate emotion, self-approval of an appropriate one.

Then, distinct from any one individual's affective representations, we have *external* to the individual a sociocultural environment that provides complex and often remarkably consistent feedback: (1) shared emotions, (2) prevailing or acceptable attitudes, (3) belief systems across the culture or subgroups in the culture, and (4) the values, ethics and morals communicated through schooling, peer groups, the examples of adult family members or authority figures, etc.

This perspective allows us to focus on affective *interactions* of the individual with the surrounding culture, without taking either the individual or the culture to be the sole level of analysis.

7. BELIEF STRUCTURES AND META-AFFECT

Affect stabilizes beliefs - human beings who feel good about their beliefs, proud of them or happy with them, are likely to continue to hold them. Belief structures are especially stable, partly because the repeated experience of one belief confirming another within the structure offers something to feel good about. But the psychology of belief is much more complicated than this. Beliefs, in turn, establish *meta-affective contexts* for the experience of emotion connected to the beliefs. Stable belief structures are comfortable, but this is not the same as saying they are pleasant. They may simply reinforce defenses against pain and hurt, unconsciously helping believers to feel a bit better about themselves. Such meta-affect can be strong enough to ensure that even careful, intelligent, rational believers will find warrants to retain their beliefs in the face of apparently contravening evidence or experience.

Let us consider, as an example, the value placed by the school culture on speed and accuracy of computation in school mathematics - specifically, arithmetic and algebra. Let us consider the related beliefs that these characteristics are good indicators of mathematical ability and potential, and that they are appropriate and sufficient measures of how well mathematics has been learned. (Just to be clear, I do not myself hold these beliefs.)

The context surrounding these values and beliefs may evoke personal emotions that range from pride and pleasure (in some students) to frustration and anger (in others). The resulting changing states of feeling, initially local, form meaningful affective pathways that encode cognitive information (e.g., regarding the individual's likelihood of mathematical problem-solving success). As such pathways become better established and interwoven with cognition, a meta-affective context for doing mathematics is created in the individual - and individual beliefs are constructed (for instance, about the person's own mathematical ability, or about the nature of mathematics as a system of applied rules) that serve to support and sustain the meta-affect.

A computationally successful child in elementary school, experiencing pride or pleasure in such activity, might come to believe in himself as mathematically able, and believe that speed and accuracy are indeed good measures of mathematical ability. If he is also able, spontaneously, to construct reasonably insightful internal mathematical representations, he may come to believe that training for speed and accuracy not only suffices to separate the mathematically able like himself from his less able classmates, but also lays an appropriate foundation for the more abstract, non-computational mathematics that requires intellectual effort later on in high school. In a culture where competitiveness among young boys is encouraged, this boy feels personally validated. A belief structure is formed, establishing a comfortable meta-affective context in which the student himself is hardworking as well as talented, a *deserving* member of an achieving elite. Even if he initially dislikes computational activity, he may come to see it as a kind of necessary pain, a rite of passage, or enjoy the competitive thrill of being better than the others - so that his pride of later achievement is enhanced.

The belief structure just described, and the satisfying emotions associated with it, may be experienced by the student as *socially appropriate,* at least in academic environments. Perhaps he later becomes a teacher of mathematics for whom speed and accuracy are central student objectives. Thus we have some of the makings of one part of a belief system widely (but far from universally) held in the mathematics community.

In contrast, a computationally less successful child, experiencing frustration and pain, might come to have a lot at stake in believing herself to be "not very good at mathematics". Although she may spontaneously generate insightful visual and spatial representations, find interesting patterns, and notice logical connections in nonroutine problems, these do not translate immediately into computational speed and accuracy. The rules of computation, to be followed without thinking about them, may be seen as derived from authority, not from reasoning. In a culture where competitiveness among young girls is not encouraged, she needs a way to assert her personal worth in the context of negative affective feedback. The belief in her own lack of ability *excuses* the self-perceived failure, to herself and to others (as she represents their evaluations of her). It is not her fault she is "slow"; it is simply her lack of ability. She may not feel exactly good about the frustration and disappointment she experiences with mathematics, but she need not feel so bad about it either. There is no need for shame. She has a belief that accounts for her performance, creating a meta-affective context that is reasonably comfortable. The belief contributes to her developing self-identity as a "non-math" person. She and others may take this as enhancing her attractiveness.

For this student, too, it has become important to see mathematics as consisting of computational activity, and speed and accuracy as valid measures of ability. Her belief structure has important elements in common with the belief structure of the first student, though their self-concepts in relation to mathematics are radically different. This student's growing belief structure may also influence the kinds of warrants she later accepts for particular beliefs in mathematics – e.g., appeal to authority or social acceptability, in place of mathematically illustrative examples, diagrams, and rational arguments. The reason is *not* that she is fundamentally unable to understand mathematical reasoning. Rather, the comfortable meta-affect created by her belief in her own relative mathematical ineptness does not permit her any pleasure in the *experience* of mathematical reasoning. She prefers the safety and security of the computational rules. Perhaps she later becomes an elementary school teacher for whom speed and accuracy in mathematics are central student objectives.

Of course these stories involve highly stereotyped characters, but the ingredients of their beliefs and feelings are real-life.

8. CONCLUSION

In considering how individuals develop, we must note that *prevailing belief structures in relation to mathematics are powerfully stabilized by meta-affect.* Such beliefs are unlikely to change simply because factual warrants for alternate beliefs are offered.

Mathematics educators who set out to modify existing, strongly-held belief structures of their students (e.g., future teachers) are not likely to be successful addressing only the content of their students' beliefs, or only the warrants for their beliefs. It will be important to provide experiences that are sufficiently rich, varied, and powerful in their emotional content to foster the students' construction of new meta-affect.

This is a difficult challenge indeed.

9. NOTES

[1] This chapter, partially based on joint work with Valerie A. DeBellis, is adapted from the author's presentations at the November 1999 meeting on “Mathematical Beliefs and their Impact on the Teaching and Learning of Mathematics” in Oberwolfach, Germany and at the March 2000 meeting on “Social Constructivism, Social Practice Theory and Sociocultural Theory: Relevance and Rationalisations in Mathematics Education” in Gausdal, Norway.

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