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DEVELOPING COMPLEX UNDERSTANDINGS: ON THE RELATION OF MATHEMATICS EDUCATION RESEARCH TO MATHEMATICS

ABSTRACT. This article explores some of the intellectual bases of the apparent cultural divide between the fields of mathematics and mathematics education research. The chasm is in part attributable to epistemologies or theoretical 'paradigms', fashionable in education, that dismiss or deny the integrity of fundamental aspects of mathematical and scientific knowledge. The solution offered is for the next generation of mathematics education researchers to knowledgeably and thoughtfully abandon such 'isms' in favor of a unifying scientific and eclectic approach to research.

1. ADDRESSING A CULTURAL DIVIDE

Several years ago approximately 200 mathematicians and scientists, including Fields Medalists and Nobel Laureates, endorsed an unusual open letter that was published in a leading United States newspaper in a full-page, paid advertisement. The letter requested Richard Riley, then U. S. Secretary of Education, to "withdraw the entire list of '*exemplary*' and '*promising*' mathematics curricula, for further consideration, and to announce that withdrawal to the public." It continued,

> We further urge you to include well-respected mathematicians in any future evaluation of mathematics curricula conducted by the U.S. Department of Education. Until such a review has been made, we recommend that school districts not take the words '*exemplary*' and '*promising*' in their dictionary meanings, and exercise caution in choosing mathematics programs. (Klein, Askey, Milgram, Wu, Scharlemann and Tsang, 1999)

Many mathematics educators, and some mathematicians working to improve school mathematics, responded with dismay or anger. One perspective was offered in a letter from the Chair of the Mathematical Sciences Education Board of the National Research Council, circulated to the Board and posted on the web with permission:

> The Open Letter is notable for featuring prominent mathematicians and scientists, and few, if any, teachers and educators who are knowledgeable about the curricula in question . . . I am a mathematician who is interested in the improvement of mathematics education in this country, and I am deeply convinced that the

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expertise of my professional community has a vital role to play in educational research and policy. I have done my best to achieve such involvement. But I find that the implication symbolized by the list of signers of the Open Letter to be not only wrong, but dangerous and damaging. Ironically, it does a great deal to make serious professional collaboration impossible . . . I know from conversations with leaders in the education community that they feel that, with gestures like this Open Letter, their whole professional community is under attack by the mathematicians, and that the attempts by much of that leadership to build bridges to the mathematics community have not been met with any reciprocity. (Bass, 1999)

Before and after publication of the 1999 open letter, the popularly-termed 'math wars' raged furiously in several states of the U.S.A. Active, rival web sites – 'Mathematically Correct' [`http://www.mathematicallycorrect.com`] and 'Mathematically Sane' [`http://www.mathematicallysane.com`] – (respectively) denounced or promoted reform in mathematics education, and continue to do so. For example, Mathematically Correct quoted McEwan (1998, p. 119), as follows: "Who's to blame for the math crisis? The answer to this question is very simple: The National Council of Teachers of Mathematics . . . has betrayed us." Mathematically Sane cites the recently-expressed view of Posamentier (2003), "On the one side there are the mathematics educators who believe passionately in the 'constructivist' philosophy and on the other side there is a group of conservative mathematicians who would like to see mathematics taught as it has been for the last many decades."

The depth of the public controversy strongly suggests that some fundamental divisions, misunderstandings, or cross-purposes have come about between research mathematicians and scientists on the one hand, and those engaged in mathematics education research, curriculum development, teacher education, and K-12 teaching on the other. Extensive personal conversations with colleagues in the United States and Europe confirm this opinion. But what accounts fundamentally for the serious and possibly widening gulf? What keeps some specialists in each field from hearing each other thoughtfully, or learning from those who are expert in both fields? Why is it that so many students not only fail to achieve significantly in school mathematics, but acquire a deep aversion to it – while members of the different educational communities hold each other to blame?

And why is it that mathematics education researchers seem to lack the persuasive, if not definitive, empirical evidence that could resolve current controversies in policy and practice?

Battista argued that overwhelming evidence for 'scientifically based constructivist theory' is being willfully disregarded, maintaining:

[W]e should look to scientific researchers whose specialty is research in mathematics education. As obvious as this seems, it is usually ignored by opponents of

> mathematics reform. Because they don't agree with the findings of the specialists, they seek out researchers in other areas to buttress their case. For instance, there are educational and cognitive psychologists who occasionally conduct research on the learning of mathematics. Unfortunately, they usually apply general, essentially behaviorist theories that ignore both the methods and the results of modern mathematics education research. (Battista, 1999)

However, there is an intrinsic implausibility to the claim that eminent scientists and mathematicians are systematically rejecting scientifically valid research, simply because they are uncomfortable with the findings. The present article suggests a different kind of underlying explanation.

I approach the topic of this article with the conviction that deep divisions are neither helpful nor necessary. All of us who teach mathematics seek to develop complex understandings in our students – understandings of mathematical ideas, meanings, theorems, techniques, and applications. Simultaneously we try to extend our own complex understandings. The latter takes place in a variety of ways.

Mathematicians, like their students, study established fields of mathematics with which they are personally unfamiliar. They also generate and elaborate on new constructs, make new conjectures, prove new theorems, and find new insights in, interpretations of, and relationships among existing results. Applied mathematicians, and theoretical scientists using mathematical methods, learn well-established theories and models and draw inferences from them for their domains of application. They also develop new mathematical models and new techniques of description. Empirical scientists design and interpret experimental studies motivated by novel or well-established theories, gaining new understandings of phenomena that are expressed mathematically. The culture of mathematics and science highly values success in these activities.

Mathematics education researchers and practitioners examine mathematics and its applications from the standpoint of how students learn and how understandings develop. This endeavor is not confined to a straightforward but narrow focus on pedagogical methods and their effectiveness. Educators study the (domain-specific) psychology and epistemology of mathematical learning, problem solving, teaching, and human development. They attend to individual learning and to group processes, grappling with classroom dynamics and with issues in mathematics teacher education. They are interested in social, cultural, historical, and technological influences on mathematics and its teaching, and concerned with affect and motivation as well as cognition. Practitioners of mathematics education implement current understandings as they teach in classrooms, plan school mathematics curricula, write activities for textbooks, work with pre-service or in-service teachers, or contribute to establishing educational policies and standards. The culture of mathematics education highly values mathematical achievement and meaningful learning by students.

In addition, both communities are profoundly interested in questions from the philosophy of mathematics – what mathematics is, how it has developed historically, what it means to 'do mathematics', and why mathematics works so well in scientific and other contexts.

Logically speaking these efforts to develop complex understandings, and the resulting insights, should be mutually supportive intellectually. Not infrequently, they are.

My thesis here is that the chasm that has opened is in part attributable to the long fashionableness of certain epistemologies or theoretical 'paradigms' in mathematics education, that *dismiss or deny the integrity of fundamental aspects of mathematical and scientific knowledge*. These include today radical constructivism and social constructivism, and to a lesser extent socioculturalism and postmodernism, which share the characteristic of being 'ultrarelativist' (e.g. Confrey, 1986, 1990, 2000; Ernest, 1991, 1998; Latour and Woolgar, 1986; Steffe, 1991; von Glasersfeld, 1989, 1990, 1996). They have come to replace the earlier, correspondingly dismissive but once-dominant schools of logical positivism and behaviorism (e.g. Ayer, 1946; Skinner, 1953, 1974).

The experience base leading me to focus on these questions, and to reach the conclusions described here, is rather unusual in that it includes several crossings of the 'cultural divide' between research in the mathematical sciences and the field of education. I was trained as a theoretical physicist, and did my graduate and postdoctoral work in quantum field theory. I became an assistant professor in a graduate school of education, specializing in mathematics education research at a time when 'behavioral objectives' seemed to dominate the field nationally. Later I became an associate professor in a department of mathematics, and coordinated at the same time a science teacher education program. I moved on to direct a multidisciplinary university education center, and to organize a large, successful grant-funded university/public schools partnership project for statewide systemic change, during a period when 'radical constructivism' seemed to fuel the national reform agenda. For more than 30 years I pursued simultaneous, successful research programs in mathematical physics and in mathematics learning and problem solving.

These activities involved surprising and sometimes painful social and cultural experiences. Early in my career, the effects could reasonably have been termed 'culture shock'. I became aware in the different academic communities of powerful, tacitly held assumptions, beliefs, and expectations, conflicting deeply with each other. I belonged to no 'ism' though I explored, learned from, and rejected several of them. My scientific understandings left me profoundly skeptical of the sweeping claims and changing fashions that seemed to characterize educational research, while it became equally clear to me that relatively few mathematical scientists appreciated the challenges and complexities of K-12 education. The conflicting values of the communities to which I belonged, but did not 'really' belong, posed difficult career obstacles – much that was valued by one culture was overtly derogated by the other. And the latter situation has not improved in the three decades since my first experiences of it – if anything, it is growing more severe. An acquaintance who moved several years ago from a physics department to a graduate school of education in the United States described the resulting 'culture shock' to me quite seriously as greater than what she had experienced in emigrating to America from Russia.

The different intellectual bases with which I interacted evoked in me early on a certain disorientation or disequilibrium, as well as scientific skepticism. This ultimately strengthened my commitment to understand – and where possible, to reconcile – the intellectual sources of the disagreements. Where I am critical of mathematics education research, my criticisms stem not from a desire to discredit any body of work but from a commitment to improving the field from within it. Ultimately the perspective I take in discussing the gulf between the two communities comes from having deep roots in both, the knowledge base to take an independent view, and the belief in the *fundamental compatibility of valid, scientific understandings* obtained at different levels from different theoretical perspectives.

Though this article cites selected publications by leading researchers, its perspective is based to a considerable extent on numerous personal discussions with people in the United States and abroad whose work I respect highly. These include conversations with leading mathematicians and physicists, some of whom have been dismissive of the scientific quality of research in mathematics education; with leading mathematics education researchers, some of whom have been dismissive of mathematical claims to truth and scientific claims to knowledge about the real world; with leading psychologists, philosophers, and cognitive scientists adhering to different schools; with teachers and supervisors of school mathematics; and perhaps most influentially for me, with doctoral students and recent doctoral graduates uncomfortable with the restrictions they feel are imposed by the 'paradigms' in which they are being schooled.

I hope that the essay, while unavoidably controversial, will be heard as a clear call to younger researchers for a major change of direction in the mathematics education research field. I shall argue for far greater discernment than has been exercised in the past. It is time to abandon, knowledgeably and thoughtfully, the dismissive fads and fashions – the 'isms' – in favor of a unifying, non-ideological, scientific and eclectic approach to research, an approach that allows for the *consilience* of knowledge across the disciplines. This will help lay the needed foundation for a sound intellectual relationship between the fields of mathematics education research and mathematics.

2. Sociological vs. intellectual issues

Some may highlight other reasons for a 'cultural divide'. Indeed, part of the problem is easily attributed to the sociology of groups in complex bureaucratic and political institutions. Mathematicians and scientists work in different types of organizations, and very different social environments, from those in which primary and secondary school teachers work. Typically they also belong to different university departments, divisions, or schools from those of their education colleagues, at least in much of the United States and Europe. The nature, levels, and selectivity of professional qualification, the expectations for successful performance, and the status conferred through various kinds of recognition, are very different in mathematics and in education.

Sometimes the distinct groups vie for the same scarce resources. Furthermore a certain cultural/psychological territoriality regularly asserts itself. There can be self-satisfaction – often accompanied by approval from peers – associated with the devaluation of other groups. This leads on occasion to arrogance and defensiveness, or to expressions of stereotyping, prejudice, stigmatization, and contemptuous dismissal.

However, such sociological issues are not my topic here. I acknowledge that they exist, and while they can be quite pervasive, I do not think they are themselves strong enough to generate inevitable conflict or to deserve designation as the primary causes of the schism. For both communities, and for the next generation of students at all levels, the value of achieving meaningful improvement in mathematics (and science) education is extremely high – sufficiently high to provide a powerful incentive for cooperation and meaningful collaboration. Many examples of such good collaboration can be cited, including activities in New Jersey that I have participated in and helped to organize.

My view is that there is another set of reasons for the divide – fundamental, intellectual reasons. At the root of the problem on the education side has been the willingness of some leading researchers to commit themselves to systems of belief, methodologies, educational philosophies, or epistemological schools that fundamentally *deny or dismiss a priori the very integrity of knowledge* in mathematics and science. It is not always apparent to non-mathematicians and nonscientists that they are doing this. Sometimes, however, it seems to be done consciously and intentionally, as a way of inviting controversial attention and gaining a following.

There is a counterpart tendency in the mathematical sciences community. It is to impose – with unwarranted confidence – tacit and naive models of learning that likewise *deny or dismiss the very integrity of knowledge* in the field of education. Typically this involves separating out, taking as 'given' and valuing highly certain mathematical content and discrete skills, rather narrowly defined, for which competency at a surface level is straightforwardly measured – while dismissing or disregarding the complexity of the processes through which mathematical understanding develops in students of diverse abilities and motivations.

Then the blindness to essential, core aspects of each other's disciplines becomes mutual.

The difficulties are compounded by differences in language use that result from the distinct tacit assumptions made in the different fields, further impeding productive communication. It is this *intellectual* chasm that in my view underlies many of the current problems; not merely the incidental reasons of sociology or bureaucratic culture.

In some ways the intellectual gulf is of long standing, but in the past decade it seems to have grown substantially in breadth and depth. Earlier generations of mathematicians, psychologists, and educators made important contributions to the psychology of mathematical learning, development, and problem solving that *built on* deep understandings of the bases of mathematical, scientific, and educational knowledge. Mostly this took place *without* the far-reaching dismissals, oversimplifications, and ideologies. In this category I would include, for instance, the essential work of Jerome Bruner (Bruner, 1966; Bruner, Goodnow and Austin, 1956, 1986), Robert B. Davis (1966, 1984), Zoltan Dienes (Dienes, 1963; Dienes and Jeeves, 1965, 1970), Hans Freudenthal (Freudenthal, 1991; Streefland, 1994), Jean Piaget (1967, 1970), George Polya (1954, 1962, 1965), W. W. Sawyer (1955, 1970); Richard Skemp (Skemp, 1982, 1986; Tall and Thomas, 2002), and many others. Today, however, it is difficult to read extensively in mathematics education without encountering belief systems – the 'isms' – that are in conflict not only with fundamental mathematical and scientific understandings, but also with each other.

In this article I would like to first partially characterize, in elementary language, what I mean by the essential, core 'integrity of knowledge' in mathematics, science, and education, whose denial or dismissal does so much damage. Then I shall provide some concrete examples of the dismissive epistemologies and their influence. My goal here is not to provide a philosophical rebuttal to the varieties of 'ultrarelativist' epistemology, which has been done effectively elsewhere (see references below). My purpose is to emphasize the need for an approach to mathematics education research that is *inclusive* rather than dismissive; that emphasizes, highlights, and gives insight into key constructs essential to the mathematics and to the physical sciences; and that relies on scientific methods to provide the most objective possible conclusions regarding effective teaching practices.

To avoid possible misunderstanding, I would like to emphasize two points at the outset.

First, in criticizing what I call dismissive approaches, I am *not* arguing that either educators or mathematicians should accept existing systems of thought merely because they are widely believed by others to be true. I am not arguing for the uncritical acceptance of some 'mathematical and scientific authority'. Sometimes established belief systems *deserve* to be challenged or overthrown, on rational and/or empirical grounds. For example, at one time, astrology – the reading of character or the forecasting of human events from the observed positions of heavenly bodies – was an established system to which some prominent scientists as well as many nonscientists adhered. But today there are *valid* reasons whereby the scientific community rejects, not just the specific content, but the integrity of the knowledge behind astrological character reading and forecasting. It is rightly discarded as *untrue* (though the study of the psychological reasons for the continued widespread belief in astrology is a fascinating subject).

The dismissals I challenge are those that take place on *a priori* ideological or philosophical grounds. I use the term 'ideological' to refer to frameworks that are *not open to falsification.* Especially unfortunate, and damaging, are the dismissals that occur without fundamental understandings of what is being dismissed, and without offering any improved perspective that might replace what is dismissed with better explanations.

Second, let me stress that in advocating the abandonment of 'isms' by the mathematics education community, I am *not* suggesting disregard of the empirical observations or the often valuable insights that have been reported by their advocates, and I am *not* advocating the replacement of current ideologies by an 'absolutist' or 'objectivist' ideology. Ideologies often contain 'kernels of truth', and where this is so we should preserve and build on these even as we come to understand the limitations imposed and the damage done by their dismissive aspects. In my own work, I have drawn on certain ideas important in behaviorism, empiricism and logical positivism, and structuralism, as well as radical and social constructivism, and socioculturalism – and I encourage others to draw on these ideas – but in each case, without the need to reject the admissibility or potential usefulness of other constructs.

3. INTEGRITY OF MATHEMATICAL AND SCIENTIFIC KNOWLEDGE

This section describes briefly what I mean by the 'integrity of knowledge' with respect to mathematics and empirical science. Its content is not novel. My goal is to summarize, at an accessible level, some basic ideas involved in understanding mathematics and the natural sciences that are commonly neglected in the enthusiasm for 'isms' – and to point out their central importance for mathematics education, despite some of the current fashions.

As this article is mainly addressed to an audience of educational researchers, it focuses most closely on the integrity of scientific and mathematical knowledge, toward which I argue the mathematics education field must reorient itself.

Scientific truth

Within the mathematics education community, we need a wide and deep understanding of *the rational and empirical foundations of scientific truth, validity and objectivity* – not an *a priori* rejection of these ideas.

Elementary concepts underlying science include the use of classification, comparison, and quantitative measurement to render observations communicable and, most importantly, *reproducible* (i.e., more objective). They include distinctions among assumption, observation, and inference. They include the roles of models and theories, including mathematical models; the roles of qualitative and exploratory research; and the notions of a repeatable experiment and *replicable* experimental results. They include the use of statistics to *validly* infer the generalizability of scientific findings from a sample to a population. They include the distinction between correlation and causation; the notion of a *controlled* experiment and that of a blind or double-blind experiment, and the rationales behind these. They include the idea that experimental hypotheses can be *confirmed* or *disconfirmed* empirically, with the outcome having implications for the *truth or falsity* of theories compatible with the hypotheses. They include the idea of using the quantitative 'goodness of fit' between theoretical predictions and empirical data to establish a domain of applicability for a theory.

The power of scientific methods consists precisely in our ability, through their use, to achieve descriptions and understandings of the natural world that to some degree *transcend* or *factor out* cultural bias, prevailing opinion, powerful social belief systems, and individuals' predispositions and misconceptions. These methods do *not* entail claims of 'absolute truth' or 'certain knowledge' – indeed there is an important sense in which scientific truth is by its nature tentative, and always approximate. Above all, scientific theories and the hypotheses that derive from scientific theories are in principle and practice *falsifiable.* But scientific methods *do* involve warranted claims of objectivity, predictive validity, verifiability, domains of applicability, and empirical truth. The practice of science involves reaching warranted conclusions (based on evidence, experimentation, and rational inference) about a 'real world' that predated human existence, comprised of partially and imperfectly-understood physical, chemical, and biological structures, and functioning according to partially and imperfectly-understood physical laws.

Thus we can understand how an earlier theory such as classical Newtonian mechanics, for example – known since the early twentieth century to be 'false' in our (objectively) relativistic and quantum-mechanical universe – provides such a quantitatively good account of observations of macroscopic bodies moving at low velocities. The theory remains a *valid close approximation* in a certain observational domain.

I use the term 'integrity of scientific knowledge' to describe the methods and constructs described here, because without them we have no processes for doing science, and no basis for saying that claimed knowledge is scientific or distinguishing science from pseudoscience or superstition. 'Scientific' claims made without such a basis are without foundation, without scientific integrity.

Of course, none of these ideas are or should be immune from criticism or deeper exploration. Discarding them *a priori*, however, can only contribute to the chasm. There are two extremely important reasons why mathematics educators and mathematics education researchers should understand the nature of scientific methods of inquiry, and the reasons they are so powerful, even when their own work is exclusively exploratory, qualitative, inventive, or interpretive.

First, mathematics is frequently characterized as 'the language of science'. The possibility of quantitative, reproducible measurement is one of the things making that language possible. Much of mathematics is thus descriptive of the physical world, and teaching this relation (rather than denying that it is possible to represent the real world, or to have knowledge about it) must be included as a goal of mathematics education.

Second, scientific methods of research enable us to arrive at valid, generalizable research conclusions through well-designed experiments that meet certain criteria. Mathematics education as a discipline includes aspects that are creative, interpretive, historical, and artistic, as well as aspects that are scientific. To substantiate claims for the effectiveness of teaching methods, whether 'constructivist' or not, and of mathematics curricula, whether 'reform' or 'conventional', the mathematics education community needs to be able to affirm knowledgably the applicability of scientific criteria to educational research, and strive to meet them (rather than denying the very possibility of 'objective' knowledge). In my view the absence of such a consensus, particularly among radical constructivists, accounts for some justified skepticism among scientists and mathematicians toward the "scientific constructivism" affirmed by Battista (1999, 2001). Scientific criteria, appropriately understood and interpreted, are applicable to qualitative as well as to quantitative research methods in the study of mathematical learning and problem solving (Goldin, 1997a, 2000).

A strand of ultrarelativism

Let me be concrete and provocative. I already imagine some of my education colleagues reacting negatively to my statement about astrology at the end of the previous section, and to some of the scientific ideas above. The following statements paraphrase actual conversations with prominent individuals in mathematics education or related fields.

"What gives you the right to speak of *valid* reasons for rejecting the belief systems of others? Isn't this simply a dogmatic, hegemonistic, absolutist assertion? Other world views (not unlike the objectivist, conventional science in which you were trained, and to which you continue to adhere) are just different, personally or socially constructed ways of understanding the world around us. In some cultures, they are highly valued. Science may be socially dominant today, in Western culture, and for *this* reason Western science (as distinct from astrology, for instance) is important for our students to learn so that they may participate in this aspect of our culture. But we cannot ever know for certain that science is true or an alternative false. Indeed, so-called 'scientific facts' presuppose the prior acceptance of theories and a host of tacit assumptions within which the facts are expressed. One cannot find an absolute or transcendent frame of reference. Other systems – other *experiential realities* – are just as viable for those who believe in them as science is for its believers."

Such comments are paraphrases of philosophical ideas expressed frequently in the educational research community. These ideas are rarely criticized within that community. They are *not* asserted exclusively or mainly about astrology, and they are not typically intended as arguments in support of any particular, nonscientific world view. Rather they are put forth as arguments against the absolutism or dogmatism that their proponents attribute to scientists and mathematicians, when the ideas of (objective) truth or validity enter the discussion.

Advocates of these views often combine *objective* terminology, such as 'reality', 'truth', or 'world', with *subjective* modifiers, speaking thus of 'subjective' or 'experiential reality', the individual's 'world of experience', 'truth for the individual', or 'consensual truth in the culture', as if these are the only legitimate meanings or interpretations such terms can have. 'Knowledge' is often taken to be synonymous either with 'belief', 'viable belief', 'justified belief', or 'socially shared belief' (and essentially independent of the *truth* of the belief).

The ease with which these views are expressed today reflects, in my view, the acceptance that has been accorded certain of the 'isms' – particularly (but not limited to) radical constructivism, social constructivism, socioculturalism and postmodernism.

Now the arguments themselves *contain important nuggets of validity.* They allude, for example, to the interplay between scientific facts and the theories that frame them, and to the historical fact that ideas previously accepted as scientific 'truths' have often been superseded by wholly new theories (Kuhn, 1970). They allude also to sophisticated issues in the philosophy of science. Up to a point, they invite extremely interesting and important philosophical discussions (Laudan, 1990). However, as *a priori* assertions they are (unlike scientific claims) *nonfalsifiable by evidence or reason.* They are *not explanatory* of the power of science in achieving the prediction and control of natural processes. They do not enlighten students or researchers as to the meaning of 'truth' or 'validity' as these are understood by scientists. The nonfalisifiability of such 'ultrarelativist' philosophical stances depends ultimately on the dismissal of 'rational' objections to them, by considering rationality itself to be merely an arbitrary, socially constructed system of discourse.

The comments paraphrased above exemplify the denial of the integrity of scientific knowledge with which I am here concerned. The substantive conversation with serious research scientists ends when they are put forth as underlying principles of education or educational research. But objectivity, reliability, validity, empirical verifiability, truth, and of course falsifiability, stand at the center of what knowledge in the natural sciences is about. When these are dismissed, the chasm between science and education becomes virtually unbridgeable.

It is because of the absence of a rational and evidentiary base for the validity of astrology, together with the overwhelming compatibility of competing models with the scientific evidence, that the integrity of astrological 'knowledge' is *appropriately* rejected.

There remain many open problems in the philosophy of science. Some of these have to do with characterizing the diverse techniques grouped together here as 'scientific methods'; with describing the strengths and limitations of the warrants for scientific claims (including limits on generalizability); and with understanding the foundations of the cross-cultural efficacy of scientific methods. But in general these problems are *not addressed at all* by asserting vigorously the cultural dependence of science, nor by highlighting the many occasions when the scientific community has succumbed to unscientific political, religious, or cultural belief systems, nor by dismissing from the start the notions of 'truth', 'objectivity', or 'validity' that pose the philosophical issues in the first place.

As a small example, consider the idea that scientific 'facts' are theory-dependent. This idea is *essential* to science. Without the atomic theory, the 'fact' that the familiar yellow metal we call gold is an element with atomic number 79 would be meaningless; thus the fact is theory-dependent. But the sweeping assertions that all science is socially constructed, or that scientists can never in principle represent the real world, do not help us the least bit in understanding the *truth* of the fact that gold has atomic number 79 (or, indeed, that it had atomic number 79 long before human beings evolved, or invented a system of numeration, or developed the periodic table of the elements). They do not help us characterize the complex interplay between scientific facts and theories, or to distinguish science from religion, politics, poetry, or any other human activity.

When central concepts of science are dismissed – based on a philosophical 'paradigm' – by those who educate teachers in mathematics (the language of science), or by those who do research on the learning of mathematics in the hope of influencing educational policies and practices, then those who engage in the dismissal to that extent *compromise* themselves as 'mathematics' educators. Furthermore, it is one thing to raise and discuss many sides of sophisticated philosophical questions about the limits of scientific truth, and how scientific ideas are expressed, with those who already understand thoroughly the methods of science and their value as a means of achieving objectivity and predictive power. In considering many of the statements by proponents of ultrarelativist 'isms' (see below), I can and do appreciate that under certain technical interpretations some interesting or useful point is being made. But it is something else entirely to deny the very possibility of scientific truth or objectivity when we are teaching children, or teachers of children, who have not the knowledge base to appreciate what scientific methods can achieve.

Mathematical truth

Still more importantly for mathematics education, ‘truth’ is a central notion of mathematics (as well as a long-standing, fascinating topic in the philosophy of mathematics). Here the concept is used to mean something different from (but compatible with) the notion of ‘empirical truth’ in science. Again my discussion here can be only at an elementary level, but it is a level appropriate to the mathematics taught in elementary schools, secondary schools, and undergraduate colleges.

The field of mathematics has been characterized by many as the study of pattern (e.g. Sawyer, 1955, 1970; Steen, 1990). This includes patterns detected in the natural world, and patterns in systems invented by human beings. Some but not all of the latter correspond to observable, real-world patterns. To study patterns in a system, mathematicians seek to characterize the system as precisely as possible. One way in which this is done is to formulate axioms or postulates that describe the system. Thus we have a collection of propositions – mathematical statements – that are taken from the outset to be ‘true’; that is, their truth is *assumed.*

However, when postulates are interpreted as *descriptive* of some concrete system or class of systems, we have a distinct but related notion of ‘truth’. The latter notion of ‘truth’ (i.e., the *applicability* of the axioms to the system under discussion) *cannot* merely be assumed. It needs to be checked.

For example, multiplication is *assumed* to be a commutative binary operation for some categories of mathematical systems. When we *interpret* multiplication as is usual for natural numbers, one can conjecture its commutative property and check it in various ways – in an elementary schoolroom, the teacher might guide the children to discover that the discrete objects in a 3×5 array can be placed in one-to-one correspondence with the objects in a 5×3 array by rotating one of the arrays. The children may then generalize from this pattern. But when we interpret multiplication as is usual for square matrices, the property is *false* (though some other algebraic properties of addition and multiplication remain true). We thus associate systems of matrices with *different* mathematical categories (e.g., non-commutative algebras, where multiplication is *not assumed* to be commutative) from commutative systems such as the natural numbers, integers, or rational, real, or complex number fields.

Further propositions that can be *derived* (i.e., ‘proven’) from a given set of assumptions, by means of well-defined rules of inference, are likewise considered to be 'true'. These are called theorems. If the rules of inference are appropriate, the theorems *also* will be descriptive of the same system or class of systems that the original assumptions describe. Thus the enduring achievement of Euclid can be understood as follows: he was able to *describe* geometric entities (points, lines, triangles, circles, and so forth) by means of relatively few *assumed* axioms and postulates, and *prove* from these many complex and sophisticated theorems using explicit rules of logical inference.

The axioms and postulates of Euclidean geometry were considered from ancient through medieval times to be intuitively obvious, or 'self-evident truths'. The rules of inference (to the extent they were precisely stated) were likewise thought to be intuitively apparent or necessary 'laws of logic'. During this period of history, mathematicians rarely distinguished the 'logical truth' of mathematical axioms and theorems from their truth as interpreted for specific systems. Likewise, in classical antiquity, little distinction was made between 'truths' of mathematics and 'truths' of the natural world. Centuries ago (but not today) mathematical truth was understood as absolute, and the most elementary truths were thought to be self-evidently so – as in the case of the Euclidean postulate that any two (distinct) points lie on one and only one (straight) line. However, it was understood also in ancient times that many of the theorems of geometry are *not obvious.* The power of geometry in particular, and mathematics in general, inheres in part in its capability of proving results concerning geometric shapes, numbers, or other still more abstract entities, that are *true* but *far from apparent.* That is, mathematics provides definite answers to questions about patterns that without its methods cannot be effectively addressed.

The notion of 'truth' in mathematics has evolved importantly over centuries of history (Kline, 1980). A major breakthrough, and a concomitant change of perspective, came with the understanding that the mathematical 'truth' of propositions is always in relation to the system of axioms within which the propositions are formulated – and that such axiom systems can be far more arbitrary than once was believed. This understanding *did not* come and *could not* logically have come from the general, *a priori* arguments of radical constructivists. It arose from specific, important logical and mathematical developments.

Consider one brief example. Euclidean plane geometry was formulated to be about idealized points, (straight) lines, and circles lying in a plane as we might ordinarily visualize them. The proposition that 'any two points lie on one and only one line' *seems* then to be an elementary statement about these entities, properly understood as we imagine them. Like the other propositions of Euclid, the proposition appears to be a 'self-evident truth'. But suppose the word 'point' is *interpreted* to mean 'pair of antipodal points on the surface of a sphere'. Suppose that 'straight line' is *interpreted* as 'great circle on the surface of a sphere'. Then the proposition becomes, 'any two pairs of antipodal points on the surface of a sphere lie on one and only one great circle,' which is also 'self-evident' (depending perhaps on one's capacity for visualization) but which 'means' something quite different. Now most of Euclid's definitions and postulates about points and lines *translate* into statements about pairs of antipodal points and great circles; they remain true in the latter interpretation as well as the former. A straight line traverses the shortest distance between two points (remaining, of course, on the surface) in either interpretation. However, the famous 'parallel postulate' does not hold for the great circle interpretation. In this interpretation, distinct 'straight lines' always intersect! We thus obtain a concrete, *Euclidean* model (embedded in three dimensions) that satisfies *non-Euclidean* geometric axioms.

The naive objections – that a straight line does not 'really mean' a great circle, and that the 'real' shortest distance between points on a sphere's surface cuts through the interior – are familiar to mathematicians introducing these ideas to students for the first time. The level of mathematical abstraction required here entails *abandoning* this notion of 'real meaning' in favor of the power of deduction from axiomatic systems whose interpretations are not fixed but variable.

Different non-Euclidean geometries, each as consistent as Euclidean geometry, were formulated by modifying Euclid's system of postulates. Algebraic systems were constructed, generalizing the complex numbers, in which multiplication is non-commutative (first the quaternions, followed later by matrix algebras and operator algebras). Multi-valued logics were invented that are as consistent as two-valued logics. So-called 'nonstandard analysis' is understood to be as valid as conventional real analysis. Examples abound in mathematics of exotic topologies that nevertheless obey conventional axioms regarding open and closed sets.

There is nothing mysterious today about these perspectives, nor in the multiplicity of mathematical examples and counterexamples. By changing conventional assumptions we obtain new mathematical categories and new mathematical objects – i.e., new kinds of patterns – whose properties are different from each other. It makes no sense in mathematics to ask which axiom system is 'really' true. It *does* make sense to ask whether a system of axioms is (internally) consistent or not, or whether a particular axiom system applies to a particular model or not.

Some mathematical systems more accurately fit measurements of the physical world than do others. The best geometries we have for describing general relativity are non-Euclidean geometries. The best algebraic systems we have for describing quantum mechanics and quantum field theory are non-commutative systems.

But these understandings come following the realization that the theorems that follow from a system of mathematical axioms do not depend on the particular entities that the axioms are imagined to be about.

It was thought in the late 19th and early 20th centuries that a complete, consistent logical foundation for this level of abstraction could be achieved through the precise and rigorous formalization of mathematical statements, axioms, and rules of inference. What was discovered through the pursuit of that program was just the opposite – Gödel (1931) rigorously proved (roughly speaking) that any mathematical system sufficiently complex to describe even the natural numbers was *necessarily* incomplete. Furthermore, the consistency of its assumptions *necessarily* could not be fully demonstrated (Nagel and Newman, 1958; Hofstadter, 1979).

In short, the notion of 'truth' in mathematics has evolved greatly as mathematical understanding has increased. 'True' and 'provable' are no longer synonyms.

Nevertheless, mathematical knowledge incorporates fundamentally the objective truth of theorems relative to axiomatic systems. Here I use the term 'objective' to express the idea that once the terms and symbols are defined, and the axioms and rules of inference are agreed upon, then the correctness of well-stated conjectures (whether they are in fact theorems, or are false, or possibly indeterminate) is in a logical sense *established.* It is no longer a matter of individual preference or interpretation, social convention, negotiation, or subjective conception.

To speak prosaically, some mathematics is conventional – but having established the conventions that allow meaningful communication, what follows from the conventions is not! Some mathematical answers and procedures are correct, and others are wrong. Some ideas about mathematics are *valid mathematical conceptions,* consistent with the system(s) under discussion, while other ideas are mathematical *misconceptions* or mistaken conceptions.

Hence it is essential in mathematics education to distinguish between *alternate conceptions and misconceptions.* The former term can appropriately refer to situations where the student tacitly adopts a nonstandard convention, or bases a conclusion (validly) on assumptions different from those expected by the teacher, or describes a different mathematical pattern from that expected by the teacher, or uses (validly) a nonstandard method to arrive at a mathematical conclusion, or brings a different intuition (conceptual or imagistic representation) to the situation from that of the teacher or that described in the textbook. The conception may then be mathematically valid, but in some way not the expected one – a *bona fide* alternate, even if it is not 'viable' in the context of negotiated mathematical assumptions or conventions. The latter term appropriately refers to situations where the student makes a logical mistake, adopts internally contradictory conventions, conceives some property of a pattern incorrectly (so that it is not true), represents a mathematical relationship invalidly, or uses an incorrect, logically inappropriate, or inapplicable method to arrive at a mathematical conclusion – even when the conception is psychologically 'viable' for the individual.

In speaking of the integrity of mathematical knowledge being denied categorically by some in the education community, I refer to this notion of mathematical truth and to the knowledge base of (objective) mathematical results, properties of systems of representation, theorems, proofs, and logically valid techniques and methods of reasoning that enable us to arrive at such mathematical truths, as well as the role of mathematics as representational of properties of the physical world.

Of course there are also deep, unresolved issues in the philosophy of mathematics. Some are of long standing, and many are suggestive of similar issues in the philosophy of science or general questions in the philosophy of knowledge. How may the notions of mathematical existence and mathematical truth be further refined logically? What accounts for the extraordinary power of mathematics in describing the natural world? What are the psychological, social, cultural, and aesthetic influences on the establishment of conventional mathematical systems, and how do they function? What is the relation between the human mind and mathematics (Changeux and Connes, 1995)?

The very notion of a 'pattern' (that the field of mathematics studies) already raises interesting questions. In what sense do patterns exist *apart* from the mind or minds construing them? To what degree should patterns be regarded as objective, subjective, or both – 'out there' (external), 'in here' (internal, subjective, and idiosyncratic), or relational? In what sense can it be said that the pattern, and truths about the pattern, existed before the conventions were established? I am not in any way proposing a return to Platonism as the answer to this important question. I am, however, suggesting that the *a priori* rejection of any such notion of 'truth' or 'existence' is not justified, and is part of the problem of the cultural divide.

The nugget of value in some of the 'isms' fashionable in education is their *emphasis on addressing the subjective component* of 'pattern' – psychologically subjective, for the individual student, and socially subjective, for the classroom culture – but unfortunately, this has been done to the exclusion of other components.

Real progress with the questions posed here *contributes to our understanding* of the nature of mathematics, including the nature of mathematical truth. It takes account of the above mathematical ideas, and more. In contrast, exponents of some of the 'isms' claim to have solved the problem by dismissing (intentionally or not) the very possibility of 'objective' mathematical truth or validity, seeming to make deeper mathematical understanding of the basis of such objective truth unnecessary. This again denies the integrity of the discipline in which we are committed to educating children and teachers of children. To that degree, those who engage in the dismissal again compromise themselves as 'mathematics' educators. This, in my view, is a core intellectual issue underlying the schism.

4. INTEGRITY OF EDUCATIONAL KNOWLEDGE

The denial and dismissal of the integrity of knowledge occur in *both* directions across the cultural divide.

A substantial knowledge base, consistent with numerous empirical quantitative and qualitative studies, has evolved from several generations of research on mathematical learning, problem solving, teaching, and development. It includes the characterization of mathematical skills, concepts, schemata, cognitive structures, and the relationships among all of these, as they are developed by individuals. It takes account of the complex and highly diverse ways in which children of various ages, levels of development, and structures of ability learn skills and construct or infer meanings, find patterns, solve problems, draw inferences, and construct tacit or explicit mathematical understandings. Among the important distinctions are those between *conceptual* knowledge and *procedural* knowledge, *tacit* knowledge and knowledge that is *overt*, *discovery* learning processes and *reception* learning processes, *internal* systems of cognitive representation and *external* or conventional representational systems, *contextualized* mathematics and mathematical *abstraction*.

Mathematics educators know something about the fundamental importance in thinking and learning processes of imagery and visualization, discourse, language and metaphor, strategies and heuristic planning, reasoning by analogy, affect, motivation and belief systems, and social and cultural contexts. There is now considerable evidence supporting the view that 'mathematical ability' is not a unitary construct, but involves many different components – and that different children learn optimally in different ways. Educational researchers appreciate the severe limitations inherent in trying to separate the 'content' of mathematics from the processes of learning and doing mathematics through which mathematical content is meaningfully acquired.

In the field of education, researchers and experienced teachers understand well that mathematics learned solely as a system of rules and contingencies is learned *nonmeaningfully* by many children, although mastery of standard algorithms and procedures remains an essential curricular goal.

These distinctions and perspectives, acquired through decades of research, are as central to the field of mathematics education today as the notion of 'truth' is to the field of mathematics. Unfortunately, some mathematicians – with a confidence that is unwarranted by their expertise – tacitly deny or dismiss the integrity of this knowledge base, preferring a much simplified view. To paraphrase such a view (based, again, on numerous personal conversations with mathematicians whom I respect), a collection of standard notations, terminology, mathematical algorithms, and problem-solving procedures are tacitly taken to *define* the content of school mathematics, and to constitute the 'material that needs to be covered.' Skills in performing these procedures rapidly and automatically are seen as the main or only important prerequisites to later conceptual understanding. Mathematical ability is often assumed to be an innate, unitary characteristic of the individual. Achievement is defined by test scores, and thus mathematical ability is ultimately to be measured by speed and accuracy on tests – that is, it refers exclusively to how rapidly the student can reproduce formal notational procedures and solve problems using them, after training and practice in them.

Thus I can hear some of my mathematical colleagues, of 'traditionalist' inclination, commenting rhetorically, "This is the trouble with education. How can you expect students to master concepts at a deep level when they cannot even perform straightforward operations? Some students just don't have the *ability* to understand the concepts behind the mathematics, or to solve non-routine problems. You're never going to get them to do it. We have to give these students at least the basic skills, and to do that there is no alternative to a lot of drill and practice. In the meantime, we shouldn't hold back the more talented students on their account. I myself learned mathematics perfectly well in a 'traditional' program. The important thing is just to focus on the content."

Again, there is a nugget of truth here. First, there are some underlying reasons – important to appreciate – that many mathematicians see mathematics as *comprised* of symbolically-written abstract systems. Formal logical reasoning is powerful when applied in such systems. Most mathematicians appreciate how in higher mathematics visual intuitions can mislead, and thus have come to trust rigorous proofs carefully formulated in symbolic notation. This creates a certain reluctance to focus too long on imagery, on mathematics 'in context', or on students' nonstandard, spontaneously constructed representations. It is consistent with placing the main emphasis on formal notational competencies. Skill in the performance of symbolic mathematical operations is indeed a necessary prerequisite to more advanced work, and such skill interacts with the development of meaningful understanding. This aspect of learning is too often undervalued or neglected by educators. And 'holding back' students who are ready and motivated to go forward is (of course) poor educational practice.

However this set of views, at least tacitly, disregards and dismisses much of the *valid* knowledge base that has developed in the mathematics education research field, a few features of which I have mentioned above. It is just as nonfalsifiable as the educational 'isms'– when 'real' mathematical achievement is defined by conventions, rules, and test scores, and mathematical ability is defined through speed and accuracy on standardized items, it is tautological that students of greater ability will achieve more. Drill and practice in items parallel to the test items will in the short term raise achievement. Recent politically-motivated suggestions in the United States to dismiss qualitative research studies in education as 'unscientific' while recognizing large-scale, quantitative studies, are themselves scientifically unsound. They are likely to skew attention toward easily-measured, short-term gains in performance on discrete mathematical skills, and away from the long-term study of meaningful and transferable understandings.

It should be clear that I do *not* maintain that most mathematicians or natural scientists concerned with education hold dismissive views. Only some do, just as only some educational researchers adhere to dismissive ideologies. But the fact that some in each field do, and that they achieve recognition from their colleagues for espousing those views, provides part of the rationale for the counterpart dismissals on the other side of the cultural divide.

5. Dismissive epistemologies

From the 1950s to the 1970s *behaviorism* was in its ascendancy in the United States, though far less so in Europe. Behaviorist psychologists rejected *a priori* all that could not be directly, empirically observed (Skinner, 1953, 1974), in accord with the 'verifiability criterion of meaning' suggested by logical positivists (Ayer, 1946). Thus they excluded not only information gained through introspection, but also any discussion of internal mental states, cognitive models, thoughts or feelings. Terms such as 'understanding' were expressly forbidden. Neo-behaviorists were somewhat more flexible, allowing 'internal responses' to serve as intermediate links between observable (external) stimulus situations and (behavioral) responses (e.g. Gagné, 1970). This exclusive reliance on that which was empirically observable allowed behaviorists to claim *scientific* validity for their dismissive stance, though examples abound in the natural sciences of constructs that have been *inferred* rather than directly observed.

Behaviorism (together with the quantitative techniques of psychometrics) fueled the 'behavioral objectives' approach to mathematics education, which fit with school accountability measures based on skills tests (Mager, 1962; Sund and Picard, 1972). By the 1970s, behaviorism was fueling the 'back to basics' counterrevolution to the 'new mathematics', which had been largely a mathematician-led movement. School curricular objectives were being rewritten across the USA to decompose them into discrete, testable behaviors (cf. Erlwanger, 1973, for a contemporary critical view) – and to eliminate the banned words. Because the pendulum has subsequently swung so far in the other direction, it is difficult for students today to apprehend how completely behaviorism came to dominate. Constructs associated with other points of view became wholly unacceptable in some educational circles, and school practices in mathematics suffered greatly.

As a school of psychology, behaviorism succumbed to the successes of theoretical ideas behind cognitive-developmental research and structural linguistics, and to the new, powerful constructs associated with the emerging field of cognitive science. These could not be formulated naturally in behavioral terms. Meanwhile the reform movement in mathematics education gathered strength in the USA through the 1980s and into the 1990s, challenging prevailing practices that had been encouraged by behaviorists.

Radical constructivists were among those contributing to the rejection of behaviorism. Thus von Glasersfeld wrote,

> For about half a century behaviorists have worked hard to do away with 'mentalistic' notions such as *meaning, representation, and thought.* It is up to future historians to assess just how much damage this mindless fashion has wrought. Where education is concerned, the damage was formidable.
>
> (von Glasersfeld, 1987, p. 10)

However, as fashions changed, many mathematics education researchers turned to new, dismissive intellectual trends that *denied the very possibility* of 'objective' truth – thus dismissing from the outset the central construct of mathematical inquiry. This denial did not derive from empirical evidence, nor did it draw strength from having made an empirically proven contribution to understanding the learning or doing of mathematics. It was, rather, a philosophical denial.

Radical constructivists excluded the very possibility of 'objective' knowledge about the real world, focusing solely on individuals' 'experiential world'. Abstract mathematical structures, apart from individual knowers, were to be rejected. 'Viability' (a subjective notion) was to replace (objective) 'validity'. Cognition and learning could *only* be regarded as adaptive responses to the individual's world of experience, and *never in principle* as reaching real-world 'truths'. Those who studied individuals' constructive processes of learning, without honoring the radical constructivists' *a priori* denials of the notions of truth, objectivity, or the possibility of representing the real world, were labeled 'trivial constructivists' or 'weak constructivists'.

> Constructivism drops the requirement that knowledge be 'true' in the sense that it should match an objective reality. All it requires of knowledge is that it be viable, in that it fits into the world of the knower's experience, the only 'reality' accessible to human reason. (von Glasersfeld, 1996, p. 310)

Radical social constructivists saw mathematical (and scientific) truth as merely negotiated social consensus, while the postmodernist trend has been to attribute such consensus to structures of power and hegemony. A perspective frequently quoted with approval is that of Feyerabend, who dismissed the possibility of scientific truth. For example Steedman (1991) quoted Feyerabend:

> . . . this appearance of success [of a scientific theory] *cannot be regarded as a sign of truth and correspondence with nature.* . . . Such a system will of course be very 'successful' not, however, because it agrees so well with the facts, but because no facts have been specified that would constitute a test and because some such facts have even been removed. Its 'success' *is entirely man-made.* It was decided to stick to some ideas and the result was, quite naturally, the survival of these ideas.
> (Feyerabend, 1968, quoted in Steedman, 1991, p. 4)

Steedman continued, "Feyerabend's respect for science, and indeed all knowledge, has led him to want science understood for what it is, rather than being misunderstood as a secularized religion." This suggestion that if one does not dismiss the ideas of 'truth' and 'correspondence with nature', one is taking science as a 'secularized religion', exemplifies the point where the chasm becomes unbridgeable.

However, my criticism of these dismissals should not be misinterpreted as devaluing the empirical study of how truth claims develop in mathematics, or the social processes involved in achieving scientific knowledge.

Interestingly, the *a priori* dismissal by the behaviorists of notions such as 'understanding' and 'acquisition of concepts' in mathematics education has recurred in the postmodernist context. In a discussion of the so-called 'paradigm wars' in education, Lerman argues for the abandonment of such words and phrases not only because they are unobservable, but because they are 'tools of regulation' (Sfard, Nesher, Lerman and Forman, 1999). The emphasis on power and hegemony relationships implies there are *moral/ideological* as well as epistemological/ideological reasons fueling the dismissals. In my view this is a troubling trend that lends additional urgency to our moving away from the 'isms.'

A detailed critique focusing on radical constructivist epistemology in mathematics education may be found in Goldin (1990), where other, earlier radical constructivist articles are also cited. Detailed philosophical critiques focusing on science and science education may be found in Nola (1998) and Kragh (1998). Written well before the 'Math Wars' had begun, the first-mentioned article highlighted the danger in radical constructivist epistemology that worthy, non-behaviorist ideas in mathematics learning and teaching "may be rendered invalid in the eyes of those who (with justification) seek an empirical, scientific basis for mathematics education research" (Goldin, 1990, pp. 39–40).

It seems to me that exactly this has now taken place. A research forum at the 24th Conference of the International Group for the Psychology of Mathematics Education attracted a large audience from many countries. Panelists and audience members reacted to a pointed question posed in a column the year before by Diane Ravitch, a former high-level government official in the U.S. Department of Education:

> ... *unlike educators* [emphasis added], physicians have canons of scientific validity ... Why don't we insist with equal vehemence on well-tested, validated education research? Lives are at risk here, too. (Ravitch, 2000, p. 75)

Unfortunately, in my view, the panel and the audience – which took a variety of stances – were very far from replying, or wanting to reply, "*Of course* educators have canons of scientific validity." There was no consensus supporting the *value*, the *feasibility*, or even the *possibility* of obtaining valid, 'objective' research findings in mathematics education (Boero, 2000; Brown, 2000; Lin, 2000). In the dominant climate of ultrarelativism, this is hardly surprising. It makes Battista's continuing effort to elucidate an empirically-based, 'scientific constructivism' (Battista, 2001) highly unlikely of success.

In my view it is also the constructivists' *exclusive* emphasis on the 'experiential world', highlighted in mathematics teacher education courses and doctoral programs, that leads inevitably to the dismissals that have helped fuel the 'math wars.' For example, Loveless (2001) quotes a letter of concern as exemplifying one aspect of the conflict:

The following statements were taken from the MathLand Teacher's Guides:
Question: "How should teachers approach this unit if they are uncomfortable with the number systems presented here?" (Gr. 5)
Answer: "The beauty of the constructivist philosophy is that it doesn't rely on the teacher as the dispenser of knowledge, but as a facilitator of experiences. In this case, the teacher should view herself as a co-learner."
...
'... there is no such thing as a number fact. There are only relationships and these relationships are created inside the child's head.'
(quoted in Loveless, 2001, p. 204)

The current climate leaves the mathematics education field susceptible to new sweeping claims of an ultrarelativist nature, that are likely to widen further the gap with mathematics. Most recently, the idea that mathematics consists entirely of 'conceptual metaphors' has been offered as the basis of a new philosophy, to be called 'mind-based mathematics', and a new discipline, to be called 'mathematical idea analysis' (Lakoff, G. and Núñez, R., 1997, 2000). Lakoff and Núñez (like Changeux in Changeux and Connes, 1995) see the human activity of doing mathematics as a domain for empirical investigation through neuroscience and cognitive science, a perspective that I share. But predictably, in the current ultrarelativist climate, their dismissive philosophical claims have attracted attention and praise in the mathematics education community. The mathematical and philosophical problem of the logical foundation of mathematics is simply and easily dismissed – there *can be* no foundation; the idea of one is just a metaphor. In mind-based mathematics (not unlike radical constructivism) there are only conceptions, no misconceptions:

The so-called misconceptions are not really misconceptions. This term as it is implies that there is a 'wrong' conception, wrong relative to some 'truth.' But Mathematical Idea Analysis shows that there are no wrong conceptions as such, but rather variations of ideas and conceptual systems with different inferential structures ... (Núñez, 2000, p. 19)

Ideas and visualizations (familiar to mathematicians) that underlie and motivate abstract constructions are renamed as metaphors, presented as if newly-discovered, and taken to *be* the mathematics. Mathematicians who might disagree are caricatured as Platonists, naive realists, or empty formalists, in disregard of the way the notion of truth has evolved. In this view mathematics cannot *possibly* 'exist' independently of human metaphors (and as we know, when something is just a metaphor, it is *not* literally true). Critiques of these views in the context of mathematics education have been published elsewhere (Goldin, 1997b, 2001).

I understand that today some variation of dismissive ultrarelativism can be a tempting response to closed-minded, apparently 'absolutist' views among mathematicians. It may seem to justify educators' openness to students' various ways of thinking mathematically, to their placing emphasis in education on the *ideas* of mathematics, on imagery and metaphor, open-ended problem solving, discovery processes, social and cultural environments, and systems of belief – all that, I strongly favor! But the ultrarelativist 'isms' *undermine* the connection between mathematics education and mathematics. They erect barriers *even to formulating* goals that should be among those central to mathematics education – conveying something of the nature of mathematical truth; demonstrating not only the power but the objectivity of valid mathematical reasoning; bringing learners to experience the processes of abstraction, generalization, precision of reasoning, and proof; as well as identifying the same abstract mathematical concepts and structures in a variety of different conceptual domains. These should be goals not only for future mathematicians, but also for the large majority of children studying mathematics.

6. Consilience of knowledge

As I have suggested in the earlier discussion, ideologies that become popular usually have important 'kernels of truth'. Behaviorism rightly rejected the prior reliance in psychology on inadequate, simplistic, or unsubstantiated mentalistic constructs as psychological explanations. It contributed to connecting psychological research to its empirical foundations. It rejected (validly) values as evidence, and called into question the broad generalizations claimed from clinical reports and anecdotal evidence.

Radical constructivism helped overthrow dismissive behaviorism, rendering not only legitimate but highly desirable the qualitative study of students' individual reasoning processes and discussions of their internal cognitions (but with the unfortunate provision that no 'objective validity' could be claimed for the conclusions of research). It led to many in-depth, observational studies that have been of value to those who have advocated meaningful, guided discovery-oriented mathematical learning.

Social constructivism pointed to the importance of social and cultural contexts and processes in mathematics as well as mathematics education, and postmodernism highlighted functions of language and of social institutions as exercising power and control.

And 'mind-based mathematics' emphasized the ubiquity and dynamic nature of metaphor in human language, including the language of mathematics.

Unfortunately, in emphasizing its own central idea, each of these has insisted on *excluding* and *delegitimizing* other phenomena and other constructs, even to the point of the words that describe them being forbidden –

including central constructs of mathematics and science – or, alternatively, certain meanings being forbidden to these words. Yet the ideas summarized here as comprising the 'integrity of knowledge' from mathematics, science, and education are not only well-known, but have proven their utility in their respective fields. There are ample reasoned arguments and supporting evidence for them.

The understandings derived from many different research approaches are necessary to mathematics education. Important and valid research findings have derived from *within* many of the 'isms'. Educational researchers who might be characterized as 'behaviorists' or 'neo-behaviorists', as well as those who might be termed 'ultrarelativists', have performed groundbreaking work. Mathematicians who might be characterized as 'Platonists' or 'formalists', as well as those holding quite different views, have achieved important mathematical insights, and argued for attention to important educational priorities.

We enhance our understandings of complex processes when mathematicians and educators who focus on different aspects of those processes are able to communicate effectively and learn from each other, not when essential distinctions are erased or the most important constructs of other disciplines dismissed. There are some hopeful signs. Despite the cultural divide, there are numerous examples of mathematicians and educators working together, in collaborative environments, on common problems. For instance, a major grant in 2002 was awarded by the U.S. National Science Foundation to the University of Georgia and the University of Michigan, for a Center for Proficiency in Teaching Mathematics involving mathematicians and mathematics educators at both institutions. One research focus is to characterize mathematical knowledge for teaching, and learn how teachers can be helped to better develop such knowledge (see Ball, 2003).

In my own studies of mathematics and science, and of the mathematical learning and problem solving processes of students, I have never found mathematical and scientific knowledge to be contradictory of educational knowledge. In my mathematics education research work, I have found the notions of 'representation' and 'representational systems' to be especially useful constructs, once the dismissive epistemologies have been bypassed. These ideas, and the ways in which they allow for the unification of many aspects that various 'isms' have emphasized and others have dismissed, are described in detail elsewhere (Goldin, 1998a,b, 2002; Goldin and Shteingold, 2001).

We need theoretical frameworks that are neither ideological nor fashion-driven. They should be such as to allow their constructs to be subject to validation. Their claims should be, in principle, open to objective evaluation, and subject to confirmation or falsification through empirical evidence. The idea that the (valid) knowledge human beings acquire in different domains (through different means appropriate to those domains) is coherent and compatible rather than contradictory, has been called *consilience* (Wilson, 1998). We expect what we learn from evolutionary biology and genetics to 'fit' with what we learn from brain science and neuroscience, and we expect the latter to 'fit' with behavioral, motivational, and social psychology. Likewise it is reasonable to expect the different useful and valid descriptions of learning emanating from cognitive science, linguistics, developmental psychology, and mathematics education research, to be *fundamentally compatible with each other.* Our knowledge bases in mathematics and the natural sciences should 'fit' easily with and augment the knowledge bases deriving from educational research in these domains.

I wish to conclude, then, by urging others doing mathematics education research to *accommodate* in their work the most applicable and useful constructs from many different approaches, but *without* the dismissals. In particular, mathematics education researchers need to understand the damage being caused by dismissive ultrarelativism, and bring to an end the uncritical acceptance of it, without resorting to other, equally dismissive perspectives. It is past time to thoughtfully reincorporate mathematical and scientific truth, objectivity, correctness, and validity, alongside other ideas, in the thinking of the mathematics education research community.

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